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Protected multiplets of M-theory on a plane wave

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Abstract: We show that the symmetry algebra governing the interacting part of the matrix model for M-theory on the maximally supersymmetric pp-wave is the basic classical Lie superalgebra SU(4|2). We determine the SU(4|2) multiplets present in the exact spectrum in the limit where μ (the mass parameter) becomes infinite, and find that these include infinitely many BPS multiplets. Using the representation theory of SU(4|2), we demonstrate that some of these BPS multiplets, including all of the vacuum states of the matrix model plus certain infinite towers of excited states, have energies which are exactly protected non-perturbatively for any value of $\mu > 0$. In the large N limit, these lead to exact quantum states of M-theory on the pp-wave. We also show explicitly that there are certain BPS multiplets which do receive energy corrections by combining with other BPS multiplets to form ordinary multiplets.

Keywords: M-Theory, M(atrix) Theories, Penrose limit and pp-wave background.

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Contents

1. Introduction

In this paper, we continue our analysis [1] of the matrix model proposed in [2] to describe M-theory on the maximally supersymmetric pp-wave [3]–[6], with hamiltonian

$$H = R\ \mathrm{Tr}\left(\frac{1}{2}\Pi_A^2 - \frac{1}{4}[X_A, X_B]^2 - \frac{1}{2}\Psi^\top\gamma^A[X_A,\Psi]\right) + \\ + \frac{R}{2}\ \mathrm{Tr}\left(\sum_{i=1}^{3}\left(\frac{\mu}{3R}\right)^2 X_i^2 + \sum_{a=4}^{9}\left(\frac{\mu}{6R}\right)^2 X_a^2 + i\frac{\mu}{4R}\Psi^\top\gamma^{123}\Psi + i\frac{2\mu}{3R}\epsilon^{ijk}X_iX_jX_k\right). \tag{1.1}$$

In our previous work, we noted that for large μ, the matrix model may be expanded about each of its classical supersymmetric vacua (corresponding to collections of fuzzy-sphere giant gravitons) to give a quadratic action with interactions suppressed by powers of $1/\mu$. In the limit $\mu = \infty$, the theory becomes free and one can explicitly diagonalize the hamiltonian to determine the exact spectrum about each of the vacua. Through explicit perturbative calculations we then estimated the ranges of parameters and energies for which perturbation theory is valid.

As we noted in [1], an intriguing feature of the matrix model is the unusual superalgebra, in which the hamiltonian does not commute with the supersymmetry generators and the anticommutator of supersymmetry generators yields rotation generators as well as the hamiltonian. This latter property opens up the possibility of BPS states in the spectrum carrying angular momenta. By examining the $\mu = \infty$ spectrum, we found that such BPS states are indeed present, and in fact there are infinite towers of BPS states annihilated by 2, 4, 6, or 8 supercharges.

The motivation for the present work is to determine which of these BPS states remain BPS (and therefore have protected energies) away from $\mu = \infty$. Typically, BPS multiplets may receive energy corrections only by combining with other BPS multiplets to form non-BPS multiplets. We wish to understand when this can happen in the present case, and therefore determine which multiplets, if any, are protected as we vary μ to finite values. By demonstrating the existence of protected multiplets, we will be able to obtain non-trivial information about the spectrum beyond the perturbative regime, and even in the M-theory limit.

We now give a concise summary of our results as we outline the paper.

In section 2, we review the symmetry algebra of the model. We show that the symmetry algebra of the SU($\mathbb{N}$) theory is a "basic classical Lie superalgebra" known as SU(4|2) whose bosonic generators are the SO(3) and SO(6) rotation generators and the hamiltonian. Since the hamiltonian is one of the generators, the spectrum of the matrix model at given values of the parameters N and μ is completely determined by which SU(4|2) representations are present. In particular, the energy of a representation (measured in units of μ) may shift as we vary the parameters only if there exist nearby representations (or combinations of representations) with the same SO(6) $\times$ SO(3) state content but different energy.

The representation theory of SU(4|2) has been studied by various authors including Kac [7] (who originally classified the basic classical Lie superalgebras) and Bars et. al. [8] (who introduced a convenient supertableaux notation which we use heavily.) In section 3, we review relevant aspects of this representation theory, and describe the complete set of finite- dimensional, positive-energy, unitary representations of SU(4|2), which we display in figure 5. We note that generic representations (called "typical") fall into one-parameter families of representations with identical SO(3) $\times$ SO(6) state content but energies which vary as a function of the parameter. In addition, there are a discrete set of "atypical" representations for which no nearby representations with the same SO(3) $\times$ SO(6) states but different energy exist.

In section 4, we describe the physical implications of this representation theory. We note that typical representations are free to have energies which vary as function of μ while atypical multiplets cannot receive energy corrections unless they combine with other atypical multiplets to form typical multiplets. We then determine all possible sets of atypical multiplets which may combine (these sets always involve only two multiplets). For a given atypical multiplet, there are at most two other atypical multiplets with which it may combine. If neither of these complementary multiplets is in the spectrum for some $\mu = \mu_0$, we may conclude that the energy of the original atypical multiplet is protected as we vary μ to nearby values. We find certain special multiplets (including those known as "doubly atypical") which cannot combine with any other multiplets to form a typical multiplet. Any such multiplets in the spectrum at some value of μ must be in the spectrum for all values of μ and the states in these multiplets have non-perturbatively protected energies. Finally, we demonstrate the exisitence of an infinite family of supersymmetric "indices" (linear combinations of occupation numbers for finite collections of atypical multiplets given in eq. (4.3)) which are exactly protected for all values of $\mu > 0$.

We then proceed to apply this knowledge to the actual spectrum of the matrix model. In section 5, we review the exact spectrum of the model for $\mu = \infty$ and describe the complete set of SU(4|2) multiplets that it contains. In section 6, we apply our representation theory reasoning to determine which states have protected energies. We first identify all doubly atypical (and therefore exactly protected) multiplets in the spectrum. These include the vacuum states plus infinite towers of excited states above each vacuum. All of these states must be present with the same energy (in units of μ), for all values of μ. Furthermore, the doubly atypical spectrum about any given vacuum has a well defined large N limit, so we conclude that these are exact quantum states of M-theory in the pp-wave background.

We then analyze the remaining atypical multiplets (which have the possibility of pairing up). We find some representations whose complementary multiplet is not present for $\mu = \infty$ and therefore cannot receive an energy shift as μ is varied from infinity. There are also pairs of multiplets in the $\mu = \infty$ that can combine to form typical multiplets. By explicit perturbative calculation we provide an example of one such pair which does receive an energy shift (and therefore must combine into a typical multiplet), as well as other such pairs of multiplets which do not receive an energy shift at leading order. As a check, we also verify a vanishing energy shift at leading order in perturbation theory for certain states that we predict are protected. Finally, based on representation theory we argue that for the single membrane vacuum, the leading perturbative energy shift for all states (including those in typical multiplets) must display cancellations leaving a result that is finite in the large N limit.

In section 7, we clarify the relation between atypical multiplets and BPS states (annihilated by one or more supersymmetry generators). We show that all BPS states lie in atypical multiplets and that all atypical multiplets contain BPS states. However, atypical multiplets generally contain some states which are not BPS. In fact, certain non-BPS atypical states carry no charges at all and yet have protected energies.

Finally, we offer some concluding remarks in section 8 and technical results in a few appendices.

Note added: after this work was completed, the paper [10] appeared which partially overlaps with section 6 of this work. For other recent work on the pp-wave matrix model, see [11].

2. Symmetry algebra

The symmetry algebra for the matrix model was discussed in [1, 2]. The bosonic generators are the light-cone translation generators P^+ (realized trivially as $P^+ = N/R$) and P^- (the matrix model hamiltonian), the SO(3) and SO(6) rotation generators M^{ij} and M^{ab}, and the creation and annihilation operators a_i, a_a associated with the center of mass harmonic oscillator. The fermionic generators include 16 simple generators q which affect the overall polarization state, as well as the 16 non-trivial generators Q which anticommute to give the hamiltonian and rotation generators.

The generators P^+, a_i, a_a and q act only on the U(1) part of the theory which decouples, as discussed in [1]. In this paper we will focus on the superalgebra generated by the remaining non-trivial generators Q, H, M^{ij} and M^{ab}, with commutation relations

$$\begin{aligned}\{Q^{\dagger I\alpha}, Q_{J\beta}\} &= 2\delta^I_J \delta^\alpha_\beta H - \frac{\mu}{3}\epsilon^{ijk}\sigma^{k\alpha}_\beta \delta^I_J M^{ij} - \frac{i\mu}{6}\delta^\alpha_\beta (\mathrm{g}^{ab})_J{}^I M^{ab} \\ [H, Q_{I\alpha}] &= \frac{\mu}{12} Q_{I\alpha}\end{aligned}$$

and additional commutators between M's and Q's appropriate to the fact that $Q_{I\alpha}$ transforms in the $(\mathbf{4},\mathbf{2})$ of $\mathrm{SO}(6)\times\mathrm{SO}(3)$.

This superalgebra satisfies all of the conditions[1] for a "basic classical Lie superalgebra," all of which have been classified by Kac [7]. As for the bosonic simple Lie algebras, these fall into several infinite series as well as a number of exceptional superalgebras. Among these superalgebras, there is precisely one whose bosonic subalgebra matches ours ($\mathrm{SO}(6)\times\mathrm{SO}(3)\times\mathrm{U}(1)_H \sim \mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{U}(1)_H$), namely the algebra $A_{3,1}$ whose compact form is known as SU(4|2).[2]

For any values of the parameters N and μ, the spectrum of the matrix model must therefore lie in (finite-dimensional[3]) representations of SU(4|2). In particular, since the hamiltonian is among the SU(4|2) generators, the energy spectrum of states for given N and μ is completely determined by which SU(4|2) representations are present. As a result, the energies of states in a given representation can only change as a function of μ if there are nearby representations with the same $\mathrm{SU}(4)\times\mathrm{SU}(2)$ state content but different energies.

In the next section, we will see that physically allowed representations of SU(4|2) come in two types, known as typical and atypical (depicted in figure 5). Typical representations lie along one-parameter families of representations which differ only by their energy eigenvalue. States in these representations can therefore shift up or down along the one parameter trajectories as a function of μ and therefore have energies which vary as a function of the parameters. On the other hand, atypical representations are isolated in the sense that there are no nearby representations with the same $\mathrm{SU}(4)\times\mathrm{SU}(2)$ state content. Physical states in these representations therefore have energies which are fixed as we vary

[1] The conditions are that the algebra G is simple, the bosonic subalgebra is reductive, and that there exists a non-degenerate invariant bilinear form on G.

[2] Actually, our hamiltonian corresponds to a non-compact U(1) generator, but this will make no difference to the representation theory.

[3] We will demonstrate this below.

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μ, except in special circumstances in which two such atypical representations combine to form a typical representation which may then shift to nearby typical representations with different energy.

By understanding the representations theory of SU(4|2), and then determining precisely which representations are present at $\mu = \infty$ where the complete spectrum is known [1], we will be able to prove that certain infinite towers of states have energies that are protected as we vary μ away from $\mu = \infty$. This will provide precise information about the spectrum of the matrix model even for small μ where perturbation theory is inapplicable and also in the large N limit defining M-theory on the pp-wave background.

We now turn to a discussion of the relevant properties of SU(4|2) representations.

3. Representations of SU(4|2)

In this section, we review various aspects of the representation theory of SU(4|2) that will be relevant to the matrix model. For a much more complete treatment, the reader is encouraged to refer to the article by Kac [7], as well as further developments in [8, 9].

The algebra SU(4|2) naturally decomposes into subspaces with specific eigenvalues for the U(1) generator (energy),

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_{-1}$$

where $\mathcal{G}_0$ is the bosonic subalgebra $\{M^{ab}, M^{ij}, H\}$ (whose generators all commute with H), and $\mathcal{G}_1$ and $\mathcal{G}_{-1}$ describe fermionic generators with positive and negative H eigenvalues $\pm\mu/12$ ($Q_{I\alpha}$ and $Q^{\dagger I\alpha}$ respectively).

Any given representation of SU(4|2) splits up into a set of irreducible representations of SU(4) $\times$ SU(2) each labelled by an energy (the eigenvalue of the U(1) generator). Acting on a given state with fermionic generators in $\mathcal{G}_1$ or $\mathcal{G}_{-1}$ leads to states in different SU(4) $\times$ SU(2) representations with higher or lower energy. Since $\{Q_{I\alpha}, Q_{J\beta}\} = 0$, our physical states will always lie in finite dimensional representations. Explicitly, if $|\psi_i\rangle$ are the states in a given SU(4) $\times$ SU(2) representation, the states

$$|\psi_i; \{\epsilon_j, \tilde{\epsilon}_k\}\rangle \equiv (Q_1^\dagger)^{\epsilon_1} \cdots (Q_8^\dagger)^{\epsilon_8} (Q_1)^{\tilde{\epsilon}_1} \cdots (Q_8)^{\tilde{\epsilon}_8} |\psi_i\rangle \qquad (3.1)$$

with $\epsilon_j, \tilde{\epsilon}_j = 0, 1$ must be a complete basis of states for the full SU(4|2) representation. To see this, note that using the commutation relations of the algebra, any product of SU(4|2) generators acting on a state $|\psi_i\rangle$ may be rearranged to a sum of states of the form (3.1) by bringing all of the $Q^\dagger$'s to the left, then all of the Qs to the left of any remaining bosonic generators. Any remaining bosonic generators acting on $|\psi_i\rangle$ simply give a linear combination $c_{ij}|\psi_j\rangle$.

The representations appearing in the matrix model spectrum must not only be finite-dimensional, but all states must have positive energies and the representation must be unitarizable (i.e. admit a positive-definite inner product). Thus, we are interested in the finite-dimensional, positive-energy, unitarizable representations of SU(4|2). We now proceed with a description of these representations.

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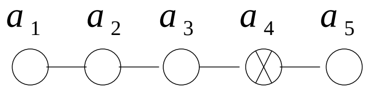

Figure 1: Kac-Dynkin diagram for SU(4|2).

3.1 Highest weight characterization of representations

The construction of representations for our superalgebra proceeds much like the familiar case of simple Lie algebras. We begin by choosing a maximal set of commuting bosonic generators from $\mathcal{G}_0$ which we denote by H_i. The remaining bosonic and fermionic generators may be chosen to be eigenvectors of the H_i whose eigenvalues we call the roots. We may divide the roots into "positive" and "negative" such that generators with positive roots together with the H_i form a maximal subalgebra. Among the positive roots, there are five "simple" positive roots which cannot be written as sums of other positive roots. Three correspond to simple positive roots of the SU(4) subgroup, one is a simple positive root for SU(2), and the final simple positive root corresponds to a fermionic generator. There exists a special choice of H_i (the Dynkin basis) with the following property. For each simple positive root α_i, we may choose generators E_i and F_i with roots α_i and $-\alpha_i$ such that

$$[H_i, H_j] = 0 \qquad [E_i, F_j] = H_i \delta_{ij} \qquad [H_i, E_j] = a_{ij} E_j \qquad [H_i, F_j] = -a_{ij} F_j$$

and the Cartan Matrix a_{ij} is given by

$$a = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 0 & 1 \\ & & & -1 & 2 \end{pmatrix}$$

Associated to this Cartan matrix is a Kac-Dynkin diagram shown in figure 1, where the fourth node corresponds to the fermionic simple positive root and in general the ith and jth nodes are joined by $|a_{ij} a_{ji}|$ lines. Note that the generators H_1, H_2, H_3 are a Cartan subalgebra of SU(4) in the Dynkin basis while H_5 is a Cartan generator of SU(2). Correspondingly, the left three nodes of the Kac-Dynkin diagram in figure 1 give the Dynkin diagram for SU(4) while the right node is the Dynkin diagram for SU(2).

In terms of this Dynkin basis, it is now straightforward to describe the finite dimensional representations. As usual, we may choose a basis of states in any representation such that the basis elements are eigenvectors of the Cartan subalgebra, and we call their eigenvalues the weights. For any finite dimensional representation, there exists a unique state with "highest weight" Λ which is annihilated by all positive roots. We will denote this highest weight in the Dynkin basis by

$$\Lambda = (a_1, a_2, a_3 | a_4 | a_5)$$

The finite dimensional representations are precisely those for which a_1, a_2, a_3, and a_5 are non-negative integers (while a_4 can be an arbitrary real number). Conversely, there exists a unique finite-dimensional irreducible representation corresponding to every such highest weight.[4]

Acting on this highest weight state with the bosonic generators, we generate an irreducible representation $V_0(\Lambda)$ of $\mathrm{SU}(4)\times\mathrm{SU}(2)$ described by SU(4) Dynkin labels (a_1, a_2, a_3) and SU(2) Dynkin label a_5 (spin $a_5/2$) with energy

$$h \equiv \frac{H}{\mu} = \frac{1}{3}\left(\frac{1}{4}a_1 + \frac{1}{2}a_2 + \frac{3}{4}a_3 + a_4 - \frac{1}{2}a_5\right) \tag{3.2}$$

The remaining states in the SU(4|2) representation are now obtained by acting on these states with negative fermionic generators. These are exactly the generators $Q_{I\alpha}$ in $\mathcal{G}_1$, all of which have positive energy $h = 1/12$, so the highest-weight representation V_0 is always the unique irreducible representation with lowest energy among the $\mathrm{SU}(4) \times \mathrm{SU}(2)$ representations in the full SU(4|2) multiplet. Calling $|\psi_i\rangle$ the states in the highest weight representation, a basis of the full SU(4|2) representation is then given by

$$|\psi_i; \{\epsilon_j\}\rangle \equiv (Q_1)^{\epsilon_1} \cdots (Q_8)^{\epsilon_8} |\psi_i\rangle$$

where $\epsilon_j = 0, 1$.[5]

If all these states are non-zero and independent, the representation is called "typical", otherwise it is called "atypical". In terms of the highest weight, Kac showed that a representation is atypical if and only if

$$\begin{aligned} a_4 \in\ &\{a_5 + 1, a_5 - a_3, a_5 - a_3 - a_2 - 1, a_5 - a_3 - a_2 - a_1 - 2\} \\ &\cup\{0, -a_3 - 1, -a_3 - a_2 - 2, -a_3 - a_2 - a_1 - 3\} \end{aligned} \tag{3.3}$$

In the case that a_4 coincides with an element of each of the two sets on the right, the representation is known as "doubly atypical", and has additional special properties which we will describe later.

For typical representations, it is clear that we will have $\mathrm{SU}(4) \times \mathrm{SU}(2)$ representations at 9 equally spaced energy levels, and the dimension of the complete representation is

$$\dim(V(\Lambda)) = 2^8 \cdot \dim(V_0(\Lambda)) \, .$$

Atypical representations always have fewer than 9 energy levels, and the dimension is less than the right-hand side of this formula. Explicit dimension formulae for atypical representations will be given in figure 4.

[4] In general, for certain highest weights of this type, there may also exist representations which are not fully reducible, however these representations are not physically relevant since the realization of the superalgebra in the matrix model is in terms of hermitian generators.

[5] This follows immediately from the discussion above and the fact that $Q^{\dagger I\alpha}|\psi_i\rangle = 0$ for states in the lowest energy representation.

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Among these finite dimensional irreducible representations, only a subset are compatible with unitarity and positive energy. These unitarizable representations have been characterized in [9]. The conditions for unitarity may be given simply in terms of the highest weights as

$$\begin{aligned} a_4 &\in [a_5+1, \infty) \qquad & a_5 &\geq 1 \\ a_4 &\in \{0\} \cup [1, \infty) \qquad & a_5 &= 0 \end{aligned} \tag{3.4}$$

In other words, among SU(4|2) representations with a given lowest energy SU(4) × SU(2) irrep, any representation whose energy is greater than the highest energy atypical representation is unitarizable. In addition, for representations in which the lowest energy SU(4) × SU(2) irrep has trivial SU(2) part ($a_5 = 0$), there is an additional atypical representation with lower energy that is unitarizable. These conditions are depicted in figure 5.

To summarize, the physically allowed representations are those corresponding to highest weight $\Lambda = (a_1, a_2, a_3|a_4|a_5)$ such that a_1, a_2, a_3, and a_5 are non-negative integers and a_4 satisfies the conditions given in eq. (3.4).

3.2 Tensor representations and supertableaux

In the case of ordinary Lie algebras, all finite dimensional representations may be obtained as tensor products of certain fundamental representations. This is not true for superalgebras (since the representation labels include a continuous parameter), however tensor representations will play a special role in our analysis of the Matrix model spectrum (all representations at $\mu = \infty$ are of this type), so we discuss them now. For more details, see the discussion by Bars et. al. [8].

To describe the fundamental representation, we note that the superalgebra SU(4|2) may be represented by matrix generators of the form

$$\mathcal{H} = \begin{pmatrix} A & \theta \\ \theta^\dagger & B \end{pmatrix}$$

where A and B are hermitian 4×4 and 2×2 matrices with $\mathrm{Tr}\, A = \mathrm{Tr}\, B$ and θ is a 4×2 matrix of complex Grassman numbers (arbitrary linear combinations of the generators $Q_{I\alpha}$). Then the vector upon which this matrix acts defines the fundamental representation.

We may denote such a vector by ϕ_A where the index A takes values in (I, α) where I is a fundamental index of SU(4) and α is a fundamental index of SU(2).

$$\phi = \begin{pmatrix} \phi_I \\ \phi_\alpha \end{pmatrix}$$

It is clear that the states ϕ_I and ϕ_α have opposite statistics since they are exchanged by the fermionic generators. In our discussion, we will take ϕ_I to be bosonic and ϕ_α to be fermionic, but the other choice leads to an equivalent set of representations.

To determine the highest weight corresponding to this representation, we note that the matrix forms of the Cartan generators in the Dynkin basis described above are

$$\begin{aligned} H_1 &= \mathrm{diag}(1, -1, 0, 0, 0, 0) \\ H_2 &= \mathrm{diag}(0, 1, -1, 0, 0, 0) \end{aligned}$$

$$
\begin{aligned}
H_3 &= \mathrm{diag}(0,0,1,-1,0,0) \\
H_4 &= \mathrm{diag}(0,0,0,1,1,0) \\
H_5 &= \mathrm{diag}(0,0,0,0,1,-1)
\end{aligned}
$$

so the U(1) generator measuring energy is

$$
h = \mathrm{diag}\left(\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{6},\frac{1}{6}\right)
$$

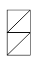
while the SU(4) and SU(2) subalgebras correspond to traceless generators in the upper left and lower right blocks respectively. Under the bosonic subalgebra, the fundamental representation thus splits into the $(\mathbf{4},\mathbf{1}) = (1,0,0)\times(0)$ representation of $\mathrm{SU}(4)\times\mathrm{SU}(2)$ with energy $h = 1/12$ and the $(\mathbf{1},\mathbf{2}) = (0,0,0)\times(1)$ representation with energy $h = 1/6$. The former representation, of lower energy, is the highest weight representation V_0, so we may immediately deduce that $(a_1,a_2,a_3) = (1,0,0)$ while $a_5 = 0$, and using the relation (3.2) with $h = 1/12$ we find $a_4 = 0$. Thus, the fundamental representation is

$$
\Lambda = (1,0,0|0|0)
$$

Higher tensor representations may be formed just as for $\mathrm{SU}(\mathbb{N})$, by considering objects with multiple A indices symmetrized in various ways.

At this point we should note that as for $\mathrm{SU}(\mathbb{N})$, there is also an anti-fundamental representation which may be obtained as the complex conjugate of the fundamental representation and has highest weight $(0,0,0|0|1)$. Unlike the $\mathrm{SU}(\mathbb{N})$ case, the anti-fundamental representation here cannot be obtained from tensor products of the fundamental representation since there is no invariant tensor analogous to the epsilon tensor for $\mathrm{SU}(\mathbb{N})$. Thus, to obtain the most general tensor representations for $\mathrm{SU}(4|2)$, we must include both fundamental and anti-fundamental indices. However, it turns out that all tensor representations involving anti-fundamental indices either contain negative energy states or are not unitarizable,[6] so we will not consider them here. Thus, henceforth when we refer to tensor representations, we will mean tensor representations with only fundamental indices.

Since the various ways of symmetrizing the indices in a tensor representation are labelled by representations of the permutation group, we may label tensor representations of $\mathrm{SU}(4|2)$ by Young tableaux, which we will call supertableaux following [8]. We use slashed boxes to distinguish them from ordinary tableaux which we will use to describe the SU(4) and SU(2) subgroups. For example, an object with two antisymmetrized super-indices $\phi_{[AB]}$ is denoted by the supertableau



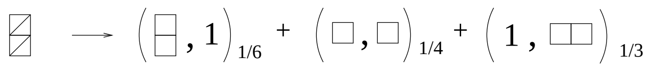
[6]This will follow from the discussion below. We should note, however, that while states in the matrix model must be in positive-energy unitary representations, physically interesting operators may be in more general representations. Indeed, the superalgebra itself is in the adjoint representation, a tensor representation corresponding to one fundamental and one anti-fundamental index.

Figure 2: Allowed tableaux and highest weights for SU(4) and SU(2).

The decomposition of a given tensor representation of SU(4|2) into individual SU(4)×SU(2) representations corresponds to the possible ways of assigning the super-indices A_i either to SU(4) or SU(2) fundamental indices. The energy of a given SU(4) × SU(2) irrep in the decomposition is given by

$$h = \frac{1}{12} n_4 + \frac{1}{6} n_2 \tag{3.5}$$

where n_4 and n_2 are the number of SU(4) and SU(2) indices respectively. For example, the tensor $\phi_{[AB]}$ decomposes into SU(4) × SU(2) tensors $\phi_{[IJ]}$ with energy $h = 1/6$, $\phi_{I\alpha}$ with energy $h = 1/4$ and $\phi_{(\alpha\beta)}$ with energy $h = 1/3$. Note that since the SU(2) indices are fermionic, the antisymmetrization of AB becomes symmetrization of $\alpha\beta$. It is convenient to represent this decomposition in terms of tableaux



The decomposition of more general SU(4|2) representations into SU(4) × SU(2) representations may be efficiently carried out pictorially as explained in [8].

The restrictions on the allowed tableaux for SU(4) and SU(2) lead to restrictions on the allowed supertableaux for SU(4|2). Recall that for SU(ℕ), we may have no more than N antisymmetrized indices, since a fundamental index can take N possible values. Thus, the maximum height of an SU(ℕ) tableau is N. Also, since N antisymmetrized indices may be contracted with an invariant epsilon tensor to give a scalar, there is an equivalence between tableaux which allows one to eliminate any columns with N boxes. The allowed tableaux for SU(4) and SU(2) are depicted in figure 2 along with the corresponding highest weights in the Dynkin basis.

The condition on SU(4|2) supertableaux is that there must exist at least one way to decompose the supertableau into allowed SU(4) and SU(2) tableaux. Pictorially, the condition is simply that the third column must have no more than four boxes. Otherwise, any decomposition would yield either an SU(4) tableau with height greater than 4 or an SU(2) tableau with height greater than 3. The most general allowed supertableau for SU(4|2) is depicted at the top of figure 4.

It is straightforward to determine the highest weight corresponding to a given supertableau. Note that from eq. (3.5), the lowest energy SU(4) × SU(2) representation for a given supertableau is obtained by assigning as few indices to SU(2) as possible. This is accomplished by associating only those boxes which are below the fourth row to SU(2), with the complete set of boxes in the first four rows forming the SU(4) tableau. Since

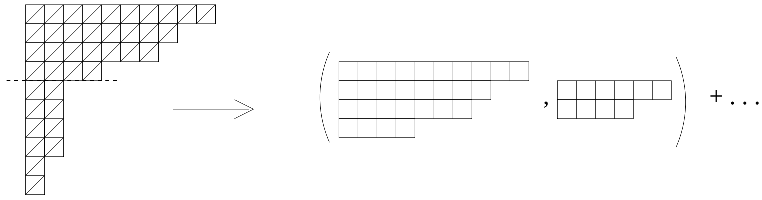


Figure 3: Lowest energy SU(4) × SU(2) multiplet in the decomposition of a general SU(4|2) multiplet.

the SU(2) indices are fermionic, the SU(2) tableau is obtained from the part of the supertableau below the fourth row by a flip on its diagonal, thus exchanging symmetrization and antisymmetrization. This decomposition is depicted in figure 3.

Given this lowest energy SU(4) × SU(2) representation V_0, the highest weight components (a_1, a_2, a_3) and (a_5) are the Dynkin labels for the SU(4) and SU(2) representations in V_0. To determine a_4, we may use the formula (3.2) where the energy h is given by eq. (3.5). In this case, n_4 and n_2 are the total number of boxes in the supertableau in the first four rows and in the remaining rows respectively.

The general allowed supertableau and the corresponding highest weight are depicted at the top of figure 4. We note that there is an equivalence between supertableaux given by adding k columns with four boxes and subtracting k rows with two boxes, or in the notation of figure 4,

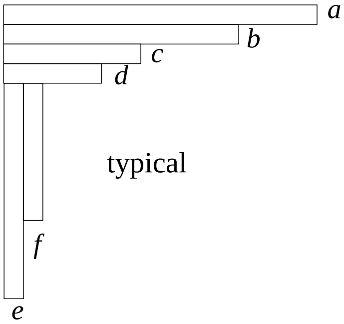
$$(a, b, c, d, e, f) \to (a + k, b + k, c + k, d + k, e - k, f - k) \tag{3.6}$$

where k is any integer such that the resulting supertableau is sensible. By this equivalence, any tensor representation may be denoted uniquely using a tableau with no more than one box in the fifth row.

Comparing the highest weight with the conditions for atypicality, we find that the atypical tensor representations are exactly those corresponding to tableau whose second column has no more than three boxes, while the the doubly atypical representations correspond to those whose first column has no more than two boxes. The general supertableaux for singly and doubly atypical representations are also depicted in figure 4 along with their highest weights. Also, using the methods in [8], we have computed the dimensions for each type of tensor representation and included these formulae in figure 4. From [8], one may also compute the relative number of bosons and fermions in these representations, and one finds that that they are equal for typical and singly atypical representations, but different for doubly atypical representations, with an excess of $a - b + 1$ bosonic states, where a and b are the number of boxes in the first and second rows of the doubly atypical tableau.

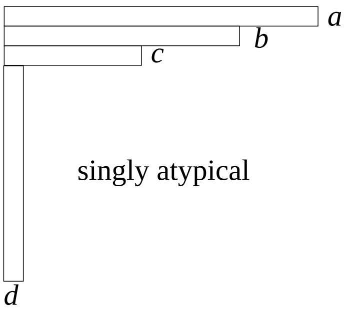
We may now compare the highest weights in figure 4 with the conditions for positive energy and unitarity in eq. (3.4). We find that every tensor representation is positive energy and unitarizable, and further, that every positive-energy unitarizable finite-dimensional irreducible representation with integer highest weight is a tensor representation.[7] This is

[7]Since general tensor representations involving both fundamental and anti-fundamental indices also



$$\Lambda = (a-b, b-c, c-d \,|\, d+e \,|\, e-f)$$

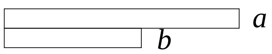
$$\dim = \frac{64}{3}(a-b+1)(b-c+1)(c-d+1)(e-f+1)(a-c+2)(b-d+2)(a-d+3)$$



$$\Lambda = (a-b, b-c, c-1 \,|\, d \,|\, d-1) \qquad d \neq 0$$

$$\Lambda = (a-b, b-c, c \,|\, 0 \,|\, 0) \qquad d = 0$$

$$\dim = \frac{8}{3}(a-b+1)(b-c+1)(a-c+2) \cdot (4abcd + 2abc - 2abd + 2acd + 6bcd + 2ac + 4bc - 2ad - 4bd + 2cd - a - b + 3c - 3d - 3)$$



doubly atypical

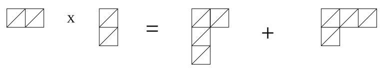
$$\Lambda = (a-b, b, 0 \,|\, 0 \,|\, 0)$$

$$\dim = \frac{1}{3}(a-b+1)(4a^2b^2 + 4a^2b + 12ab^2 + 2a^2 + 10b^2 + 8ab + 4a + 4b + 3)$$

Figure 4: Supertableaux, highest weights, and dimensions for general tensor representations of SU(4|2)

summarized in figure 5 which displays the complete set of physically allowable representations along with the discrete subset corresponding to tensor representations and the further subset corresponding to atypical representations.

Finally, we note that tensor products between tensor representations may be computed from the supertableaux with the usual Littlewood-Richardson rules for computing tensor products of SU($\mathbb{N}$) representations. In this case, when multiplying two tableaux, we keep only the resulting tableaux which are allowed tableaux of SU(4|2). For example,



correspond to integer highest weights, they must have negative energy states or be non-unitary, as asserted above, since all of the positive-energy unitarizable integer-weight representations are tensor representations involving only fundamental indices.

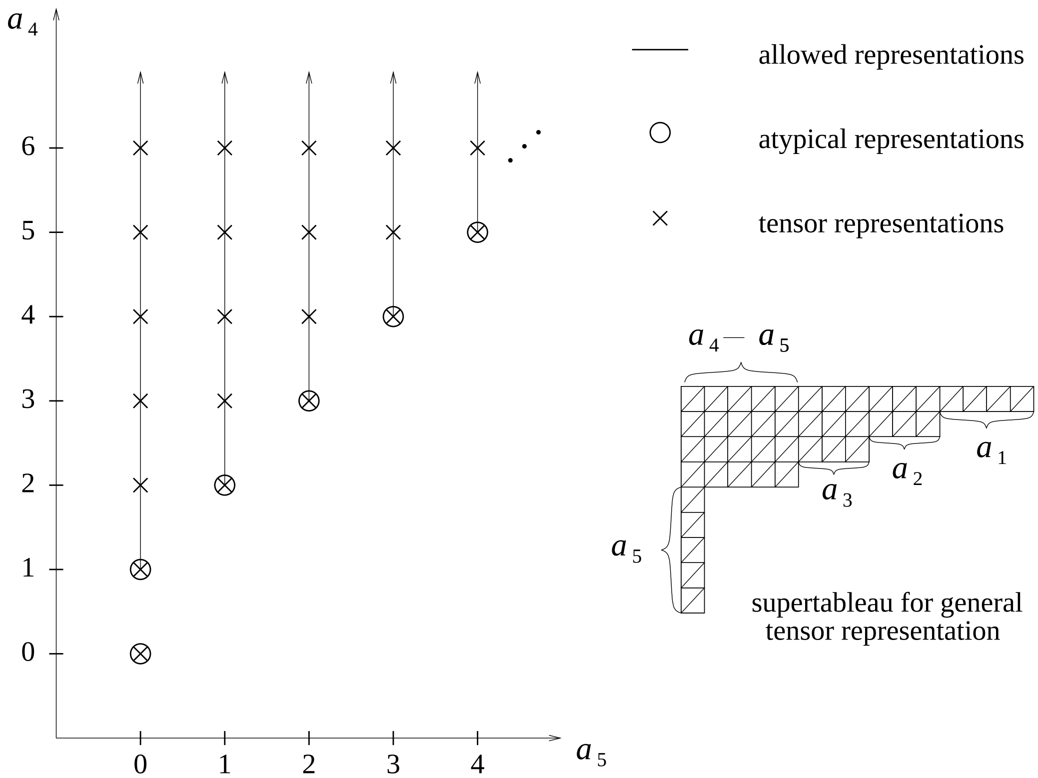


Figure 5: Physically allowed representations of SU(4|2) for a given (a_1, a_2, a_3)

4. Physical implications

We have shown that for the matrix model describing M-theory on the maximally supersymmetric pp-wave, the symmetry algebra generated by the 16 non-trivial supercharges Q, the SO(3) and SO(6) rotation generators M, and the hamiltonian H is the basic classical Lie superalgebra SU(4|2). In the previous section, we reviewed the physically allowable representations of SU(4|2) and showed that these consist of one-parameter families of typical representations as well as a discrete set of atypical representations, as depicted in figure 5. The typical representations in a one-parameter family (corresponding to the vertical lines in the figure 5 excluding the lower endpoints) have identical SU(4) $\times$ SU(2) content and differ only by the overall energies of the states (the energy differences in a representation do not change). On the other hand, atypical representations have no nearby representations with the same SU(4) $\times$ SU(2) content but different energy.

For any fixed values of the parameters N and μ, the spectrum of the matrix model will consist of some discrete set of these SU(4|2) representations which may include typical and atypical representations. We can denote this information by a set

$$S = \{\Lambda_i\}(N, \mu)$$

of the highest weights corresponding to the various representations. We would now like to understand what can happen to this spectrum of representations as we vary the parameter μ.

In general, the $\mathrm{SU}(4) \times \mathrm{SU}(2)$ quantum numbers of any given state can't change, and the energy may only vary continuously. For states in any typical representation, the energy may shift up or down (with the same shifts for all states in the representation) as we vary μ since there are nearby representations with the same $\mathrm{SU}(4) \times \mathrm{SU}(2)$ content with both higher and lower energy. This corresponds to continuously changing some of the highest weights in our set S along the one-parameter families of figure 5. On the other hand, states in atypical representations generally cannot receive any shift in energy since, there are no representations nearby in energy with the same $\mathrm{SU}(4) \times \mathrm{SU}(2)$ state content. However, it is possible that two (or more) atypical representations could combine into a typical representation or vice versa. This would correspond to a discontinuous transition in S in which certain atypical highest weights appear or disappear. We will now determine for which representations this is possible.

Let $\mu = \mu_0$ be a point for which such a discontinuous transition occurs. That is, we assume that the states in a set of multiplets at μ arbitrarily close to μ_0 arrange themselves into a different set of multiplets when μ reaches μ_0. Among the states involved in such a discontinuous transition, there will be some (or possibly more than one) state $|\phi\rangle$ of highest weight. The weight of this state (which must change continuously as we vary μ) must be present as a highest weight in the set S both at μ_0 and away from it, but by assumption, the $\mathrm{SU}(4) \times \mathrm{SU}(2)$ content of the $\mathrm{SU}(4|2)$ representation in which $|\phi\rangle$ sits will change as μ changes from μ_0. From figure 5, we see that the only case in which the $\mathrm{SU}(4) \times \mathrm{SU}(2)$ content of a representation changes with an infinitesimal change in the highest weight is the situation in which a highest weight corresponding to a typical representation reaches the bottom of its one parameter family. At this point, the corresponding representation becomes atypical and will have less states than the nearby typical representation from which it arose. For the transition to be possible, the remaining states which drop out of this representation must form a separate representation (or representations) of $\mathrm{SU}(4|2)$ with some lower highest weight(s).

To see if this is possible, we consider a general one parameter family of typical representations corresponding to highest weight $\Lambda = (a_1, a_2, a_3|a_5 + 1 + \epsilon|a_5)$ where $\epsilon > 0$. These representations all have the same $\mathrm{SU}(4) \times \mathrm{SU}(2)$ content with energies depending linearly on ϵ. Let $A = \{(R_i, h_i)\}$ be the $\epsilon \to 0$ limit of this set of $\mathrm{SU}(4) \times \mathrm{SU}(2)$ representations and energies. Similarly, we will have some set B of $\mathrm{SU}(4) \times \mathrm{SU}(2)$ representations and energies in the atypical representation with highest weight $(a_1, a_2, a_3|a_5 + 1|a_5)$. The set B will be some subset of the set A, and we may define a set C to be the elements of A not in B. We then check whether the elements in C match with some set of complete $\mathrm{SU}(4|2)$ representations.

It turns out that in every case, the elements of C precisely correspond to the decomposition of a single atypical representation (whose highest weight may be found by finding the lowest energy representation in C). For $a_5 > 0$, the states of C match exactly with the representation of highest weight $\Lambda = (a_1, a_2, a_3 + 1|a_5|a_5 - 1)$, while for $a_5 = 0$, the states of C match exactly with the representation of highest weight $\Lambda = (a_1, a_2, a_3 + 2|0|0)$. Using the dimension formulae in figure 4 it is straightforward to verify that $\dim(B) + \dim(C) = \dim(A)$ for the representations given.

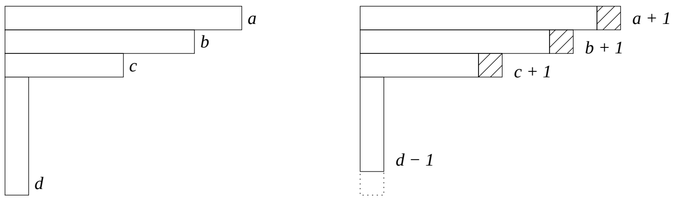


Figure 6: Atypical representations which can combine into a typical representation

Figure 7: Atypical representations which can never combine

We may conclude that the complete set of possible discontinuous transitions is

$$(a_1, a_2, a_3|a_5 + 1 + \epsilon|a_5) \longleftrightarrow (a_1, a_2, a_3|a_5 + 1|a_5) \oplus (a_1, a_2, a_3 + 1|a_5|a_5 - 1) \tag{4.1}$$

for $a_5 > 0$ and

$$(a_1, a_2, a_3|1 + \epsilon|0) \longleftrightarrow (a_1, a_2, a_3|1|0) \oplus (a_1, a_2, a_3 + 2|0|0)\,. \tag{4.2}$$

with $a_5 = 0$. In terms of the supertableaux, the pairs of atypical representations which can combine to form a typical representation are those for which one of the tableaux has one less box in the first column and one more box in each of the first three rows, as depicted in figure 6.

Thus, if a certain atypical representation exists in the spectrum at some value of $\mu = \mu_0$, it may only receive an energy shift for nearby values of μ if a complementary typical representation, obtained by the operation in figure 6 or its inverse, is also present in the spectrum at $\mu = \mu_0$, and any such energy shift must be positive.

4.1 Exactly protected representations

There are certain representations, depicted in figure 7, which are not members of any pair of representations which can combine.

In particular, these include all doubly atypical representations (i.e. those corresponding to tableaux with less than three rows).[8] We may conclude that if the spectrum contains any of these representations for some value of μ, then they must be present for all other values of μ, and the energies of states in these representations are protected from any perturbative or non-perturbative shifts.

4.2 A family of supersymmetric index theorems

We will now show that in addition to the occupation numbers for each type of multiplet in figure 7, there are additional exactly protected quantities involving the atypical multiplets not in figure 7.

[8]Actually, it is obvious that doubly atypical representations cannot combine with other representations to form typical representations since they all have more bosons than fermions while all singly atypical and typical representations have the same number of bosons and fermions.

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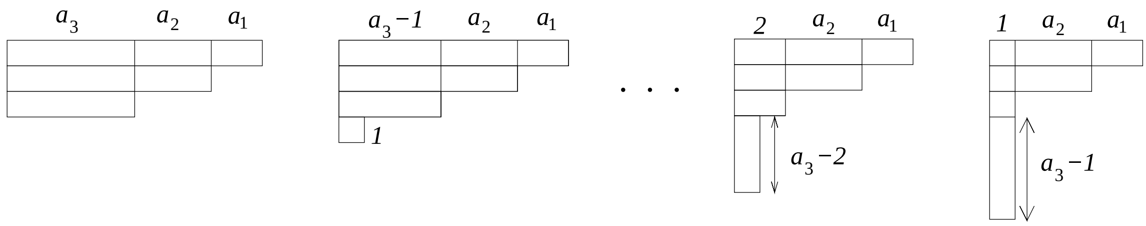


Figure 8: Chains of atypical representations in which nearest neighbors may combine

We first note that all atypical multiplets may be arranged into finite chains as depicted in figure 8, where moving to the left in the chain corresponds to performing the operation in figure 6. The leftmost multiplet in each chain corresponds to a tableau with less than four rows, while the rightmost multiplet in every chain corresponds to a tableau with less than two boxes in the third row. Each atypical multiplet will appear in exactly one chain. The chains may be labelled by the elements (a_1, a_2, a_3) of the highest weight for the leftmost multiplet (i.e. the highest weight of its lowest energy SU(4) representation), and the length of each chain will be the greater of a_3 and 1.

Now, consider some chain of atypical multiplets $(A_1, \ldots, A_n)$. By definition, each pair (A_i, A_{i+1}) may combine and shift along some one parameter family of typical multiplets which we denote by T_i. From the discussion above, if A_i is present in the spectrum at $\mu = \mu_0$, it will be protected as μ is varied away from μ_0 as long as neither of its nearest neighbor multiplets A_{i-1} or A_{i+1} in the chain appear in the spectrum for $\mu = \mu_0$. However, this multiplet is not necessarily protected for all values of μ. The reason is that one of its neighbor representations may appear in the spectrum at some $\mu = \mu_1$ without introducing a new A_i multiplet if a representation in T_{i+1} or T_{i-2} shifts to the bottom of its one-parameter family and splits. For the T_{i+1} case, we would then have representations A_i, A_{i+1}, and A_{i+2} all present at $\mu = \mu_1$. As μ is varied further, it is then possible for A_i and A_{i+1} to combine and shift to a representation in T_i, leaving only A_{i+2}. Thus, the fact that a multiplet is protected in the neighborhood of some point μ_0 does not imply that it is protected for all μ.

While the number $\mathcal{N}_i$ of multiplets A_i in the spectrum can change as we vary μ, the set $\{\mathcal{N}_i\}$ cannot change arbitrarily. Taking into account all possible ways in which the multiplets A_i/T_i can combine/split as we vary μ, it is apparent that the most general possible change in the occupation numbers between two values of μ is a transformation

$$(\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_{n-1}, \mathcal{N}_n) \to (\mathcal{N}_1 + n_1, \mathcal{N}_2 + n_1 + n_2, \ldots, \mathcal{N}_{n-1} + n_{n-2} + n_{n-1}, \mathcal{N}_n + n_{n-1})$$

where n_i are integers such that all values on the right-hand side are non-negative. There is precisely one linear combination of the occupation numbers invariant under all such transformations, namely the alternating sum

$$\mathcal{I} = \sum_k (-1)^k \mathcal{N}_k \,.$$

Therefore, this quantity defines a "supersymmetric index" which is exactly protected for all values of μ.

In summary, given any chain of representations labelled by the SU(4) highest weight (a_1, a_2, a_3) of its leftmost multiplet, there exists an exactly protected quantity $\mathcal{I}(a_1, a_2, a_3)$ given by the alternating sum of occupations numbers of multiplets in the chain. Explicitly, we have

$$\begin{aligned}\mathcal{I}(a_1, a_2, a_3) = \mathcal{N}(a_1, a_2, a_3|0|0) - \sum_{n\geq 0} \mathcal{N}(a_1, a_2, a_3 - (2n+2)|2n+1|2n) + \\ + \sum_{n\geq 0} \mathcal{N}(a_1, a_2, a_3 - (2n+3)|2n+2|2n+1)\end{aligned} \tag{4.3}$$

where $\mathcal{N}(\Lambda)$ denotes the number of multiplets in the spectrum with highest weight Λ.

We would now like to apply what we have learned to the actual physical spectrum of the matrix model, which we determined for $\mu = \infty$ in [1]. By identifying which representations are present for $\mu = \infty$ we will be able to use the results of this section to see which states are protected as μ moves away from ∞.

5. SU(4|2) representations in the matrix model spectrum at $\mu = \infty$

In [1], the exact spectrum of the matrix model was determined in the $\mu = \infty$ limit. From this explicit construction, we will now determine which SU(4|2) representations are present in this limit. In section 6, we will then use the results of section 4 to investigate which of these multiplets have protected energies as we move away from $\mu = \infty$.

To begin, we briefly recall the construction in [1]. For a given N and general non-zero μ, the matrix model contains a collection of isolated vacua corresponding to the various ways of dividing up the total DLCQ momentum N into some number of distinct gravitons, which in this background appear as concentric giant graviton spheres. The radius of each fuzzy sphere is proportional to the number of units of momentum it carries. We may expand the matrix model action about any of these vacuum states, and for large μ, we find a quadratic hamiltonian with interaction terms suppressed by powers of $1/\mu$. Thus, in the limit of large μ, the various vacua become superselection sectors each described by a quadratic hamiltonian which may be diagonalized explicitly to yield towers of oscillators that generate the spectrum. The spectrum of oscillators for the single-membrane vacuum and for a general vacuum are reproduced here in tables 1 and 2.

To determine which representations will be present in the spectrum, we first note that like the hamiltonian, the supercharges Q expanded about a given vacuum become quadratic in the large μ limit and are comprised of terms containing one creation operator and one annihilation operator. For the single membrane vacuum, we find[9] (more details are given

[9]The expression for a general vacuum is identical except that the oscillators carry indices k, l corresponding to which block they arise from, and the spin sums are those given in table 2.

in the appendix A)

$$
\begin{aligned}
Q_{I\alpha} = & \sum_{j=1}^{N-1} i\sqrt{\frac{2\mu}{3}} a_{j-1\ m} \begin{pmatrix} -\sqrt{j-m}\ (\chi^{\dagger})_{I}{}^{j-\frac{1}{2}\ m-\frac{1}{2}} \\ -\sqrt{j+m}\ (\chi^{\dagger})_{I}{}^{j-\frac{1}{2}\ m+\frac{1}{2}} \end{pmatrix}_{\alpha} \\
& + \sum_{j=1}^{N-1} i\sqrt{\frac{\mu}{3}} \mathrm{g}^{a}_{IJ} a^{a\ \dagger}_{jm} \begin{pmatrix} -\sqrt{j+m}\ (\chi^{J})_{j-\frac{1}{2}\ -m-\frac{1}{2}} \\ -\sqrt{j-m}\ (\chi^{J})_{j-\frac{1}{2}\ -m+\frac{1}{2}} \end{pmatrix}_{\alpha} \\
& + \sum_{j=0}^{N-1} i\sqrt{\frac{\mu}{3}} \mathrm{g}^{a}_{IJ} a^{a}_{jm} \begin{pmatrix} \sqrt{j-m+1}\ \eta^{\dagger\ J}_{j+\frac{1}{2}\ m-\frac{1}{2}} \\ -\sqrt{j+m+1}\ \eta^{\dagger\ J}_{j+\frac{1}{2}\ m+\frac{1}{2}} \end{pmatrix}_{\alpha} \\
& + \sum_{j=0}^{N-1} i\sqrt{\frac{2\mu}{3}} b^{\dagger}_{j+1\ m} \begin{pmatrix} \sqrt{j+m+1}\ \eta_{I}^{j+\frac{1}{2}\ -m-\frac{1}{2}} \\ -\sqrt{j-m+1}\ \eta_{I}^{j+\frac{1}{2}\ -m+\frac{1}{2}} \end{pmatrix}_{\alpha} .
\end{aligned}
\tag{5.1}
$$

Thus, under the action of the superalgebra on a given eigenstate in the $\mu = \infty$ limit, the total number of oscillators is preserved. As a result, the subsector of states with any given number of oscillators (which we will sometimes loosely refer to as "particles") must arrange into some set of complete SU(4|2) representations.

To determine the representations in the spectrum, we will first determine the representations corresponding to the individual creation operators upon which the spectrum is built. The representations making up the rest of the spectrum will be obtained by tensor products of these single oscillator representations.

5.1 Single membrane vacuum

We begin by determining the single-oscillator representations for the single-membrane vacuum using table 1.

We will use the fact that physically allowed representations of SU(4|2) are completely determined by the energy and SU(4) × SU(2) representation of their lowest energy component. The lowest energy state in table 1 is the U(1) oscillator x_{00}, whose SU(4) × SU(2) representation and energy are given by

$$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix} , 1 \right)_{1/6}$$

it is easy to check that this matches with the lowest energy component of the SU(4|2) representation described by the tableau

$$\begin{smallmatrix}\boxslash\\\boxslash\end{smallmatrix} \longrightarrow \left(\begin{smallmatrix}\square\\\square\end{smallmatrix} , 1 \right)_{1/6} + \left(\square , \square \right)_{1/4} + \left(1 , \square\square \right)_{1/3}$$

We see that the remaining states in this representation match exactly with the single particle states corresponding to $\eta_{1/2}$ and β_1, which are the remaining U(1) oscillators, i.e. all these modes are proportional to the identity matrix. Thus, the U(1) oscillators form a single representation with highest weight $(0, 1, 0|0|0)$.

Type	Label	Mass	Spins	SO(6) $\times$ SO(3) Rep	Degeneracy
$S0(6)$	x^a_{jm}	$\frac{1}{6}+\frac{j}{3}$	$0 \le j \le N-1$	$(6, 2j+1)$	$6(2j+1)$
$S0(3)$	α_{jm}	$\frac{1}{3}+\frac{j}{3}$	$0 \le j \le N-2$	$(1, 2j+1)$	$(2j+1)$
	β_{jm}	$\frac{j}{3}$	$1 \le j \le N$	$(1, 2j+1)$	$(2j+1)$
Fermions	χ^I_{jm}	$\frac{1}{4}+\frac{j}{3}$	$\frac{1}{2} \le j \le N-\frac{3}{2}$	$(\bar{4}, 2j+1)$	$4(2j+1)$
	$\eta_{I\ jm}$	$\frac{1}{12}+\frac{j}{3}$	$\frac{1}{2} \le j \le N-\frac{1}{2}$	$(4, 2j+1)$	$4(2j+1)$

Table 1: Oscillators for the single membrane vacuum

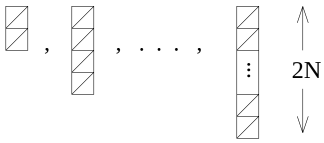
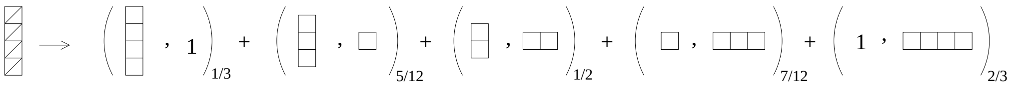


Figure 9: Supertableaux for all single oscillator multiplets for the single membrane vacuum

Apart from these states, the next lowest energy single-oscillator state is the energy $h = 1/3$, SU(4) $\times$ SU(2) singlet state corresponding to α_{00}. From figure 4, we find that the SU(4|2) representation corresponding to this V_0 is



with highest weight $(0, 0, 0|1|0)$. From table 1, we find that the remaining four components of this representation must correspond to the oscillators $\chi_{1/2}$, x_1, $\eta_{3/2}$, and β_2, which are precisely the oscillators in the SU(2) theory.

Continuing in this way, we find that the remaining oscillators arrange into SU(4|2) representations with highest weight $(0, 0, 0|2j+1|2j)$, depicted in figure 11, where the individual SU(4) $\times$ SU(2) representations in the figure correspond to α_j, $\chi_{j+1/2}$, x_{j+1}, $\eta_{j+3/2}$, and β_{j+2} respectively.[10]

Thus, in the matrix model with $P^+ = N/R$, the creation operators for the single-membrane vacuum make up the N SU(4|2) tensor representations given by the super-tableaux of figure 9.

Using the super-index notation, we may thus combine all of the oscillators from table 1 into a set of "super-oscillators"

$$a^\dagger_{[A_1 A_2]}, a^\dagger_{[A_1 A_2 A_3 A_4]}, \ldots a^\dagger_{[A_1 \cdots A_{2N}]} \,.$$

The SU(4|2) representations corresponding to states with multiple oscillators are those obtained by acting with arbitrary combinations of these super-oscillators on the Fock-space vacuum. The complete set of representations for the single membrane vacuum in the SU($\mathbb{N}$) theory is therefore given by the tensor product in figure 12. Here, we have indicated

[10]These are exactly the oscillators which couple to matrix spherical harmonics $Y_{j+1\ m}$ of spin $j+1$.

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Type	Label	Mass	Spins	SO(6) $\times$ SO(3)
$S0(6)$	$(x^a_{kl})_{jm}$	$\frac{1}{6}+\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert \le j \le \frac{1}{2}(N_k+N_l)-1$	$(6,2j+1)$
$S0(3)$	α^{jm}_{kl}	$\frac{1}{3}+\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert -1 \le j \le \frac{1}{2}(N_k+N_l)-2$	$(1,2j+1)$
	β^{jm}_{kl}	$\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert +1 \le j \le \frac{1}{2}(N_k+N_l)$	$(1,2j+1)$
Fermions	$\chi^{I\,jm}_{kl}$	$\frac{1}{4}+\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert -\frac{1}{2} \le j \le \frac{1}{2}(N_k+N_l)-\frac{3}{2}$	$(\bar{4},2j+1)$
	$\eta^{jm}_{I\,kl}$	$\frac{1}{12}+\frac{j}{3}$	$\frac{1}{2}\lvert N_k-N_l\rvert +\frac{1}{2} \le j \le \frac{1}{2}(N_k+N_l)-\frac{1}{2}$	$(4,2j+1)$

Table 2: Oscillators for general vacua

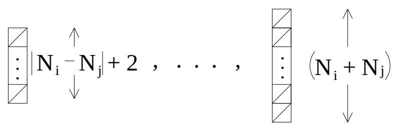


Figure 10: Supertableaux for single oscillator multiplets in a general vacuum

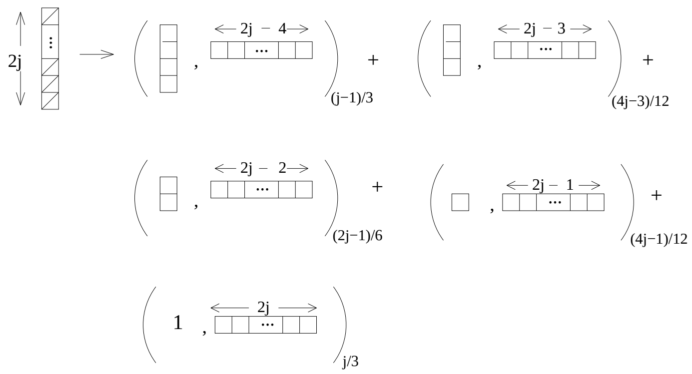
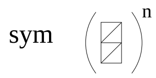


Figure 11: Decomposition of the general single oscillator multiplet

that with multiple copies of similar oscillators, we must keep only representations in the symmetrized tensor product of these oscillators. Such tensor products may be computed easily for example using the group theory calculation software LiE [12].

In the tensor product of figure 12, we have not included a factor corresponding to the U(1) oscillators. In our analysis of which states are protected, we may work directly with the representations in the SU($\mathbb{N}$) part of the theory, since the representations corresponding to the free U(1) part of the theory cannot change as we vary μ. The possible representations coming from the U(1) part (the exact spectrum of SU(4|2) representations in the U(1) theory) are given by



and this turns out to include exactly one of each allowed supertableau with an even number of boxes in each column. From now on, we will ignore this U(1) part of the wavefunction and focus on the spectrum in the SU($\mathbb{N}$) theory.

For the simplest case of SU(2), the representations are given by first factor in the tensor product of figure 12. The supertableaux corresponding to the one, two, and three oscillator states are given in figure 13.

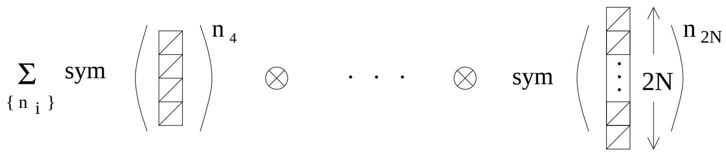


Figure 12: All SU(4|2) multiplets in the single membrane vacuum for SU(ℕ).

It is straightforward to determine the explicit oscillator expressions for these representations. For example, the highest energy states in the two-oscillator representations are the linear combinations of the states $\beta^\dagger_{2m}\beta^\dagger_{2\tilde{m}}|0\rangle$ with spin 4, 2, and 0, respectively. The other states in these representations may be determined explicitly by acting with the supercharge given in (5.1).

For the general case of SU(ℕ), we have displayed all representations corresponding to tableaux with ten or less boxes in figure 16, with labels denoting how they arise in the tensor product of figure 12.

5.2 General vacua

The analysis for general vacua is very similar to that of the single membrane vacuum. A general vacuum corresponds to collections of M_i coincident membranes at radii corresponding to momentum N_i. The oscillators for a general vacuum were described in [1, section 5.3] and are indicated in table 2.

For each ordered pair (N_i, N_j) of momenta (including the case $N_i = N_j$), we have a set of oscillators similar to those for the single membrane vacuum but which are $M_i \times M_j$ matrices and which have spins ranging from roughly $|N_i - N_j|/2$ to $(N_i + N_j)/2$. From the table 2, is straightforward to show that these correspond to supertableaux shown in figure 10.

As explained in [1, section 5.4], physical states satisfying the Gauss law constraint are obtained by acting with traces of arbitrary products of the matrix oscillators on the Fock-space vacuum. (Of course the product is required to give a square matrix in order to take a trace.)

To work out the SU(4|2) representations corresponding to these physical states, we may define super-oscillators

$$(a^\dagger_{[I_1\cdots I_{|N_i-N_j|+2}]})_{M_i\times M_j}, \ldots, (a^\dagger_{[I_1\cdots I_{(N_i+N_j)}]})_{M_i\times M_j}$$

The set of representations may then be determined from the tensor product of tableaux corresponding to an allowed product of traces of these super-oscillators. One must be careful to keep only representations in the tensor product which survive any possible symmetrization, for example, the cyclic symmetry of the trace. Also, for finite N one should keep in mind relations between traces of large numbers of matrices and products of lesser traces so as not to overcount representations.

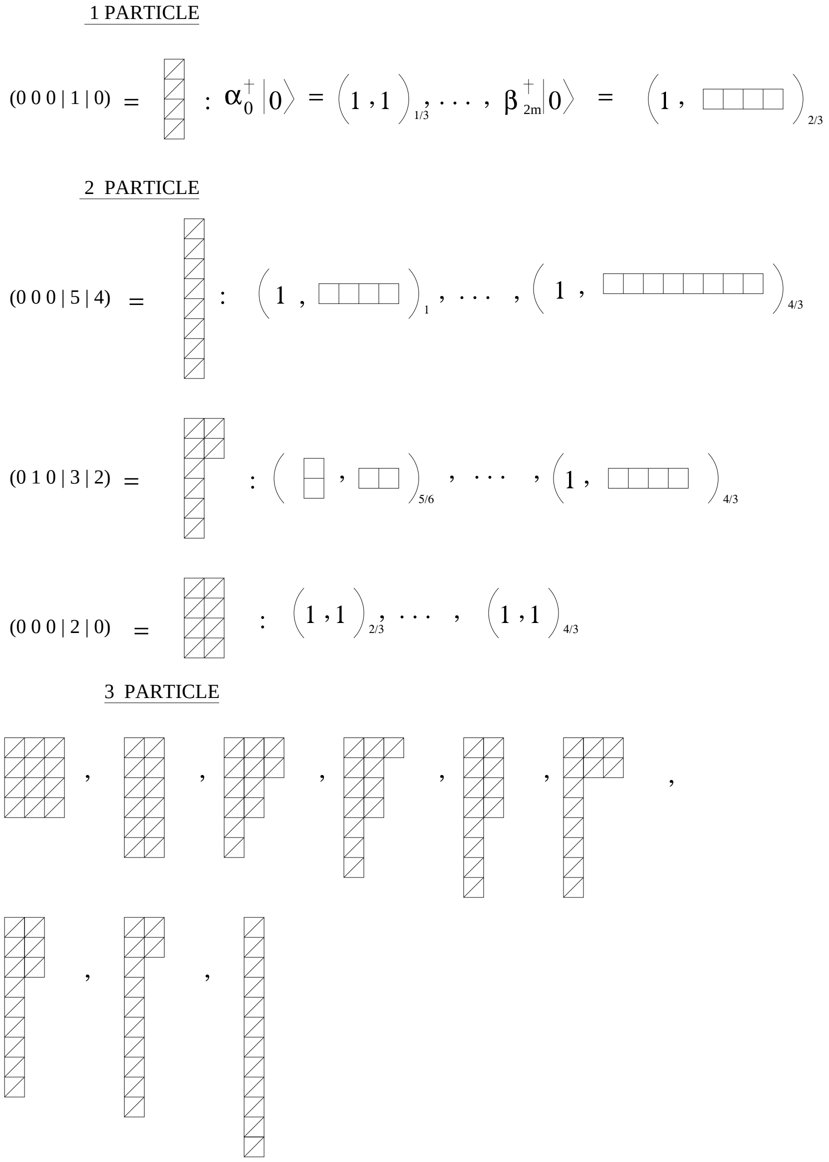


Figure 13: One, two and three particle multiplets of the SU(2) theory in the single-membrane vacuum.

Example: $X = 0$ vacuum. As an example, we consider the case of the $X = 0$ vacuum, where $M_1 = N$ and $N_1 = 1$. Here, we have only a single matrix super-oscillator

$$(a^{\dagger}_{[IJ]})_{N\times N}$$

so the representations in the spectrum will be obtained by taking products of traces of powers of this oscillator. We will ignore the U(1) part of the theory, so we assume that the oscillator $a^{\dagger}$ is traceless. In the large N limit (where there are no trace relations), the complete spectrum of the $X = 0$ vacuum is given by the tensor product of figure 14 in which *sym* indicates a totally symmetrized tensor product *cyc* indicates the cyclically symmetrized tensor product.

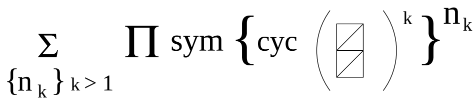


Figure 14: Spectrum of the $X = 0$ vacuum for $N \to \infty$

For finite N, one must take into account trace relations. As an example, for SU(2) we may decompose traceless matrices as $A = A_i \sigma^i$. Then using

$$\mathrm{Tr}(AB) = 2A_iB_i \qquad \mathrm{Tr}(ABC) = 2i\epsilon^{ijk}A_iB_jC_k$$

it is easy to see that the trace of any even number of matrices may be written in terms of products of traces of pairs of the matrices, while traces of any odd number of matrices may be written as products involving a single trace of three matrices and a number of traces of pairs of matrices. Thus, the spectrum for the SU(2) case is obtained by restricting the sum in the tensor product of figure 14 to $n_3 = 0, 1$ and $n_i = 0$ for $i > 3$.

The spectrum of representations involving three or less oscillators will be the same for any N, and this is depicted in figure 17.

6. Protected states in the matrix model

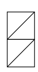
Having specified the representations present in the spectrum at $\mu = \infty$ we would now like to see which states are protected from receiving energy corrections.

6.1 Exactly protected representations

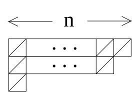
From the discussion in section 4, we concluded that all SU(4|2) representations in figure 7, which include all doubly atypical representations, have no possibility of combining with other representations to form typical representations and therefore have energies that are completely protected from all perturbative and non-perturbative corrections. We will now determine all representations in the matrix model spectrum of the type in figure 7.

We first note that all of the vacuum states, which lie in trivial representations of SU(4|2) are doubly atypical and therefore are exact vacuum states quantum mechanically for all values of N and $\mu > 0$.

To see which other exactly protected representations exist in the matrix model spectrum, note that they may only arise from tensor products of oscillators in the representation



All other oscillators correspond to tableaux with heights of at least four boxes, and tensor products involving these can never result in a tableau with a height of two or three boxes. The only representations in figure 7 that can arise from tensor products of this two-box representation are the singly atypical representations



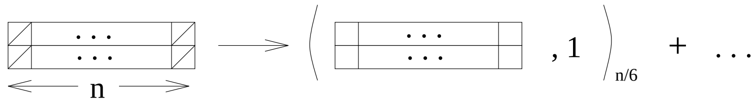


Figure 15: Doubly atypical representations in the matrix model spectrum

which we will discuss last, and the doubly atypical representations shown in figure 15 arising from products of n super-oscillators a_{IJ} which are completely symmetrized. Note that the lowest energy $\mathrm{SU}(4) \times \mathrm{SU}(2)$ representation in the decomposition of these doubly atypical representations has trivial SU(2) part and an SU(4) representation which is equivalent to the completely symmetric, traceless n-index tensor representation of SO(6). Recalling that the lowest-energy component of the two-box representation is always an SO(6) vector oscillator with energy 1/6, we see that the doubly atypical representations are precisely those built upon states with n SU(2)-singlet SO(6) oscillators whose vector indices are contracted with a completely symmetric traceless tensor (with arbitrary $\mathrm{SU}(\mathbb{N})$ trace structure).

For example, about the $X = 0$ vacuum, we have a single $N \times N$ matrix SO(6) vector oscillator X^a with spin 0, so the doubly atypical representations are those built upon the states[11]

$$
\begin{aligned}
&S_{ab} \operatorname{Tr}(A_a^\dagger A_b^\dagger)|0\rangle\,, \qquad S_{abc} \operatorname{Tr}(A_a^\dagger A_b^\dagger A_c^\dagger)|0\rangle\,, \qquad S_{abcd} \operatorname{Tr}(A_a^\dagger A_b^\dagger A_c^\dagger A_d^\dagger)|0\rangle\,,\\
&S_{abcd} \operatorname{Tr}(A_a^\dagger A_b^\dagger) \operatorname{Tr}(A_c^\dagger A_d^\dagger)|0\rangle\,, \ldots
\end{aligned}
$$

where $A_a^\dagger$ is the matrix creation operator associated with the oscillator X^a and the tensors S are completely symmetric and traceless. Note that some of the states in this series will not be independent for finite values of N, as discussed above. As an explicit check, we have computed the leading perturbative correction (at second order in perturbation theory) for the first state on this list in appendix B and found that it indeed vanishes.

For the general vacuum with M_i spheres of radius N_i, we have one $M_i \times M_i$ matrix spinless SO(6) vector oscillator $(A_a^i)^\dagger$ for each N_i, so the doubly atypical representations about a general vacuum will be those built upon states such as

$$
S_{abcdef} \operatorname{Tr}((A_a^1)^\dagger) \operatorname{Tr}((A_b^1)^\dagger (A_c^1)^\dagger (A_d^1)^\dagger) \operatorname{Tr}((A_e^2)^\dagger (A_f^2)^\dagger)|0\rangle
$$

Note that we may have traces of single oscillators for a general vacuum since only the combination $\sum_i \operatorname{Tr}(A_a^i)$ corresponds to the U(1) part.

Finally, we turn to the singly atypical representations with three rows, depicted in the previous page. The lowest energy states in these representations, again with energy $h = n/6$, are in a trivial representation of SU(2) and an SU(4) representation $(1, n-2, 1)$ corresponding to the n-index tensor of SO(6) with all but one of the indices symmetrized. These must be states formed from n spinless SO(6) creation operators with $n-1$ of the

[11] These have a structure identical to the chiral primary operators of $N = 4$ SYM theory. This is probably since $\mathrm{SU}(4|2)$ is a subgroup of the superconformal algebra $\mathrm{SU}(2, 2|4)$ governing that theory, so we expect that the chiral primary operators of AdS/CFT lie in doubly atypical representations of this subgroup.

indices totally symmetrized and a single pair of antisymmetrized indices. Because of the antisymmetric pair, this will be non-vanishing only if at least two different types of spinless SO(6) oscillators are present (i.e. the state must involve center of mass motions of two collections of membranes at different radii). For example, the lowest energy states in the representation for $n = 2$ take the form

$$\mathrm{Tr}((A^1_{[a})^\dagger)\,\mathrm{Tr}((A^2_{b]})^\dagger|0\rangle$$

where A^1 and A^2 must be distinct square matrix oscillators of different size.

Thus we find that in addition to the vacuum states, the matrix model spectrum for $\mu = \infty$ contains infinite towers of representations of the type shown in figure 7. Based on the representation theory of SU(4|2), we may conclude that all of these representations are preserved for any value of μ and that their energies receive no perturbative or non-perturbative corrections. For any given vacuum, the spectrum of these representations has a well defined large N limit (obtained by ignoring any trace relations) so we may conclude that these states are present as quantum states in the exact spectrum of M-theory on the pp-wave background.

6.2 Singly atypical representations

In addition to doubly atypical representations, the spectrum of the matrix model contains infinite towers of singly atypical representations, for example all the single particle multiplets about the single membrane vacuum. As we discussed in section 4, these are protected from receiving energy shifts except in cases where they combine with another atypical representation to form a typical representation.

A given atypical representation may combine with at most two other atypical representations, obtained either by adding a box to each of the first three rows of the supertableau and removing a box from the first column, or removing a box from each of the first three rows and adding a box to the first column, as shown in figure 6. In cases where neither of these atypical representations appear in the spectrum at $\mu = \infty$, we may conclude that the energy of the original atypical representation does not change as we vary μ away from ∞.[12]

A weaker assertion may be made when the neither of these possible atypical representations appear as states about the same vacuum as the original atypical representation but possibly appear as states about another vacuum. In this case, we expect that the original atypical multiplet is protected from receiving any perturbative corrections, but non-perturbatively there may be an energy shift since the two multiplets about the different vacua could possibly mix and combine as μ is decreased away from ∞.

To start with, we will see which states are protected perturbatively for the single-membrane and $X = 0$ vacua and then turn to the stronger condition of non-perturbative protection.

Perturbative energy shifts: single-membrane vacuum. We begin by considering the set of atypical multiplets about the single membrane vacuum, starting with the simplest case of SU(2) (i.e. $N = 2$).

[12] If one of these representations does appear, then the energy may or may not shift.

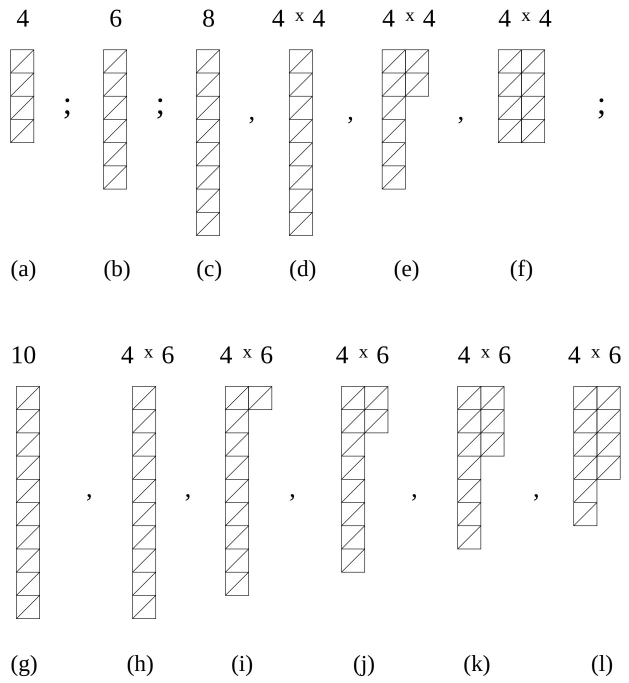


Figure 16: SU(4|2) multiplets with ten or less boxes for the SU($\mathbb{N}$) single membrane vacuum.

In this case, it is simple to argue that all atypical multiplets, such as the seven multiplets in figure 13 with less than four boxes in the second column, are protected from receiving any perturbative energy shifts. To see this, note that the spectrum of representations about the single-membrane vacuum for SU(2) is generated by tensor products of a supertableau with four boxes. Thus, the supertableaux corresponding to all representations in the spectrum will have multiples of 4 boxes. But from figure 6, we see that the number of boxes for pairs of atypical representations that can combine always differ by 2.[13] Hence, none of the atypical multiplets in the spectrum can combine and therefore all have energies which are protected from any perturbative corrections.

We now turn to the general case of SU($\mathbb{N}$). The multiplets with tableaux containing up to 10 boxes are displayed in figure 16. It is easy to check that the atypical tableaux labelled by **a**, **b**, and **e** cannot receive perturbative energy corrections since the other representations with which they could combine do not appear in the perturbative spectrum. As a check, we note that the lowest energy state in the multiplet **a** is

$$a^\dagger_{00}|0\rangle$$

whose energy shift was computed to second order in perturbation theory in [1] and found to be zero.

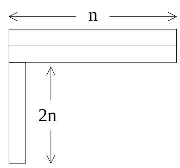
For the equivalent multiplets **c** and **d** , there is a possibility of combining with the multiplet **k** since these multiplets are related as in figure 6. Thus, we may only conclude

[13] It is important to note that while the equivalence between supertableaux given in eq. (3.6) equates tableaux whose number of boxes can differ by 2, only typical representations may have two inequivalent tableaux.

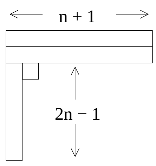
that one linear combination of the multiplets **c** and **d** is protected from receiving an energy shift. On the other hand, we have found that the leading perturbative energy shifts for all three of these multiplets is zero.[14] Thus, despite the possibility for an energy shift, we do not find one, at least to leading order.

By considering tableaux with 12 boxes one may show that the multiplets **i** and **j** are protected, while the multiplets **g** and **h** have the possibility of combining with other multiplets and shifting. Again, explicit calculation indicates that these multiplets do not shift to leading order.

To conclude our discussion of perturbatively protected states for the single membrane vacuum, we note that the following infinite tower of multiplets are protected from receiving perturbative energy corrections:



To see that these are protected, note that for a given n, this multiplet could only combine with the multiplet corresponding to the tableau



which has $4n+2$ boxes and $n+1$ columns. However, the maximum number of columns in a tableau with $4n+2$ boxes (for the single-membrane vacuum) is n, since such a tableau can arise from the tensor product of at most n oscillator representations ($n-1$ 4-box representations and a single 6-box representation).

To see explicitly which states these protected multiplets correspond to, note that one $\mathrm{SU}(4)\times\mathrm{SU}(2)$ representation in the decomposition (for which we associate all boxes in the first two rows with SU(4)) is the n-index traceless symmetric tensor of SO(6) with spin n and energy $h = n/2$. This is the multiplet corresponding to a set of n spin 1 SO(6) oscillators with spins aligned and SO(6) indices completely symmetrized, for example

$$S_{a_1\ldots a_n}(a_{11}^{a_1})^\dagger\cdots(a_{11}^{a_1})^\dagger|0\rangle$$

These have a structure which is very similar to the lowest energy states of the doubly atypical representations discussed above, except that the oscillators now carry spin. It would be interesting to determine if similar states formed out of higher spin SO(6) oscillators are also protected.

[14] This may be shown without a full calculation. First, the multiplet **d** is protected for SU(2); and for SU(ℕ), one finds that all diagrams contributing to the leading-order shift are proportional to those for SU(2) and therefore cancel. The multiplet **k** is therefore protected to leading order for SU(3) (the lowest N for which it appears) and we find that all contributions to the leading-order energy shift of **k** for SU(ℕ) are proportional to those for SU(3) and therefore cancel. The multiplet **c** is therefore also protected at leading order since it would have to have the same shift as **k**.

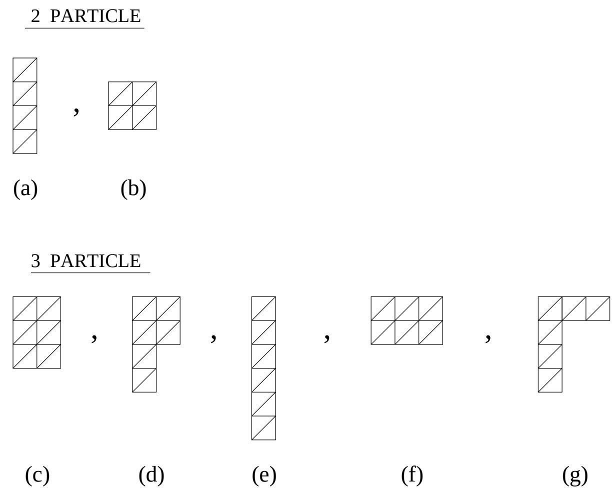


Figure 17: Two and three oscillator multiplets for the $X = 0$ vacuum

Perturbative energy shifts: X=0 vacuum. We now consider the low-energy multiplets for the $X = 0$ vacuum, depicted in figure 17. Among the two-oscillator states, the multiplet **b** is the protected doubly atypical multiplet that we have already considered (whose lowest energy state is $S_{ab}\,\mathrm{Tr}(A^\dagger_a A^\dagger_b)|0\rangle$). In addition, we have a singly atypical multiplet **a** with lowest energy state

$$\mathrm{Tr}(A^\dagger_a A^\dagger_a)|0\rangle\,. \tag{6.1}$$

From figure 6, we note that this multiplet has the possibility of combining with the atypical multiplet **c** in the three-oscillator sector whose lowest energy states are given by

$$A_{abc}\,\mathrm{Tr}(A^\dagger_a A^\dagger_b A^\dagger_c)|0\rangle$$

where A_{abc} is a completely antisymmetric anti-self-dual tensor of SO(6) (in the (0,0,2) representation of SU(4)). To see whether this occurs, we may look for an energy correction to the state (6.1) at leading order in perturbation theory. In fact, we have done this computation previously in [1] and again in appendix B and found that there is in fact a positive shift in the energy. Thus, we may conclude that the representations **a** and **c** do combine to form a typical multiplet as μ is decreased from ∞.

By considering the multiplets containing 8 boxes, we find that the singly atypical representations **d** and **e** are perturbatively protected since the representations that they could combine with are not in the spectrum. The 8-box atypical representation that can combine with the representation **g** is in the spectrum, so it is possible that **g** receives an energy shift. Finally, the representation **f** is one of the protected doubly atypical multiplets considered above.

Non-perturbatively protected states. In the previous subsections, we have shown that certain singly atypical representations are protected from receiving any perturbative energy corrections due to the absence of representations above the same vacuum with

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which they can combine. However, it is possible that pairs of atypical representations above different vacua could combine, leading to a non-perturbative shift in the energy even for states that are perturbatively protected to all orders.

By applying our group theory reasoning to the total spectrum including states above all vacua we would now like to see when this non-perturbative mixing might occur. In the process, we will also be able to strengthen our assertions for the protection of certain states from perturbative to non-perturbative statements.

For simplicity, we will focus on SU(2) whose only vacua are the $X = 0$ vacuum and the single-membrane vacuum we have already considered. The low energy multiplets for these two vacua are depicted in figures 13 and 17.

Among the four-box representations, the doubly atypical representation **b** above the $X = 0$ vacuum is certainly protected non-perturbatively, as discussed above. There are two other four-box representations, the multiplet **a** in the $X = 0$ vacuum and the identical multiplet in figure 13 for the single-membrane vacuum. Among the six-box multiplets, there is a single copy of the representation with which these can combine, so we may conclude that one combination of these representations must be protected non-perturbatively (we have already seen that one combination gets a shift).

Among the remaining six-box representations about $X = 0$, **f** is doubly atypical and hence protected non-perturbatively. As for the perturbatively protected multiplets **d** and **e**, we may now conclude that they are also protected non-perturbatively, since there are no single-membrane multiplets with which these could combine.

As a final example, we note that the perturbatively protected representation indicated by the second supertableau of figure 13 has the possibility of a non-perturbative energy shift, since the $X = 0$ spectrum contains two copies of the 10-box representation with which this can combine.[15]

Before concluding this section, we emphasise that even for states which are protected from receiving an energy shift (perturbatively or non-perturbatively) as μ is varied from $\mu = \infty$, there is a possibility that there could be a shift beginning at some finite value of μ, as explained in section 4.2. In certain cases, however, it is posible to argue using the supersymmetric indices defined in eq. (4.3), that a multiplet (other than those of figure 7) must be present in the spectrum for all values of μ.

As an example, we continue to focus on the SU(2) theory and consider the simplest non-trivial chain of the type depicted in figure 8, namely the chain with two elements corresponding (from left to right) to tableaux **c** and **a** of figure 17. It is a trivial matter to compute the corresponding index from our knowlege of the spectrum at $\mu = \infty$,

$$\mathcal{I}(0,0,2) = \mathcal{N}(0,0,2|0|0) - \mathcal{N}(0,0,0|1|0) = 1 - 2 = -1$$

Since $\mathcal{N}(0,0,2|0|0)$ must be non-negative for any value of μ we may conclude that at least one copy of the representation with tableau as in figure 17a must be in the spectrum for all values of μ.

[15]These are five oscillator states and therefore don't appear in the figure 17, but may be obtained from the general formula in figure 14.

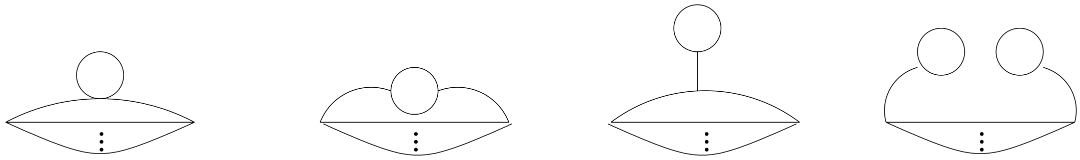

Figure 18: Loop diagrams contributing to the leading energy shifts

Thus, in various cases we have demonstrated arguments that certain multiplets are protected perturbatively to leading order, perturbatively to all orders, non-perturbatively near $\mu = \infty$, and/or non-perturbatively for all $\mu > 0$. We next consider a class of non-protected states and show that even these benefit from certain supersymmetric cancellations.

6.3 Cancellations for typical states

In [1], we noted that the leading order vanishing energy shift for the state $a_{00}^{\dagger}|0\rangle$ (which we have seen is protected) leads to cancellations in the leading order energy shifts for the non-protected states $(a_{00}^{\dagger})^n|0\rangle$ such that the non-vanishing contribution to the energy shift gives a finite result in the large N limit. In this section, we will extend this result by showing that the leading perturbative corrections (which come at second order in perturbation theory) to the energies of all states about the single-membrane vacuum have finite large N limits.

We first recall from section 6.1 of [1] that when the interaction hamiltonian is expanded about the single-membrane vacuum in terms of canonically normalized oscillators, the coupling that appears is

$$g = \left(\frac{R}{\mu N}\right)^{\frac{3}{2}}$$

which is fixed in the large N limit defining M-theory. Thus, the only possible divergent N dependence in the leading perturbative energy shifts is from the sum over intermediate states in loops. As shown in figure 18, the only diagrams which contain loops are those for which all but one of the creation operators from the initial state are contracted with annihilation operators from the final state.[16] Up to combinatorial factors, these diagrams therefore reduce to the diagrams contributing to the energy shifts of single oscillator states. It follows that all states will have finite leading energy shifts in the large N limit as long as all of the single particle states do.

The rest of argument proceeds by induction. We have already seen that all single-particle states in the four-box representation of SU(4|2) are protected and therefore have zero energy shifts in perturbation theory. Now, suppose we have proven that all single particle states up to and including those in the $2n$-box representation of SU(4|2) have finite energy shifts at leading order. To show that the single particle states in the $(2n+2)$-box representation must have a leading energy shift which is finite in the large N limit, note that these states are in an atypical multiplet which can receive an energy shift only if it combines with another atypical multiplet and becomes typical. From figure 6, the only

[16]Diagrams for which there is a single loop connected only to the states on the left or on the right cannot appear since in this case, the left and right states would not have the same tree-level energy.

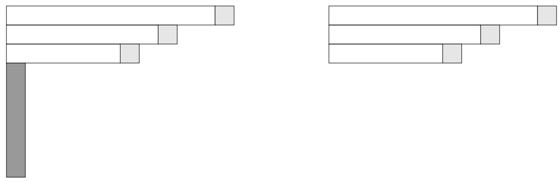

Figure 19: BPS states from atypical supermultiplets

atypical multiplet with which the single-column $(2n+2)$-box multiplet can combine is one with $2n+1$ boxes in the first column and three boxes in the second column. However, any individual oscillators contributing to the states in this multiplet must be from SU(4|2) representations with at most $2n$ boxes, so the leading perturbative energy shift for such a representation must be finite in the large N limit. It the two atypical multiplets do combine to form a typical multiplet and thus receive an energy shift, the shift for the states in the $(2n+2)$-box multiple must be identical to the shift for the states in the $(2n+1,3)$-box multiplet, and therefore must also be finite in the large N limit. This completes the inductive proof.

Hence, at least to the leading non-trivial order, perturbative corrections to the energies of states about the single-membrane vacuum are completely well-behaved in the large N limit. We will discuss this result further in section 8.

7. Relation between atypical representations and BPS states

In this paper, we have seen that the spectrum of the pp-wave matrix model must fall into representations of the superalgebra SU(4|2). This superalgbra has special multiplets which are called atypical with the property that there are no nearby multiplets with the same SU(4) $\times$ SU(2) state content but different energy. Atypical multiplets have fewer states than the typical multiplets with nearby highest weights. Further, the energies of states in atypical multiplets are protected unless the atypical multiplet combines with another atypical multiplet to form a typical multiplet.

It is clear that many of the properties of atypical multiplets are the same as properties usually associated with BPS multiplets in more familiar supersymmetry algebras. In the usual case, a BPS state is defined to have the property that it is annihilated by one or more of the hermitian supersymmetry generators, and all states in a BPS multiplet share this property. In [1], we showed that the matrix model contains infinite towers of states which are BPS in this usual sense. It is natural to guess that these BPS states will be associated with atypical multiplets, however, the precise connection between BPS states and atypical multiplets certainly requires clarification, which we now provide.

We will show that all atypical multiplets contain BPS states and all BPS states belong to atypical multiplets. However, not all states of an atypical multiplet are BPS. Indeed, there are atypical states carrying no charges at all that still have protected energies!

We begin by recalling the discussion of BPS states in [1]. Starting from the relation

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H - \frac{\mu}{3}\left(\gamma^{123}\gamma^{ij}\right)_{\alpha\beta} M^{ij} + \frac{\mu}{6}\left(\gamma^{123}\gamma^{ab}\right)_{\alpha\beta} M^{ab} \tag{7.1}$$

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and choosing a set of Cartan generators M^{12}, M^{45}, M^{67}, and M^{89} for SO(6) $\times$ SO(3), we showed that the eigenvalues of $\langle\psi|\{Q_\alpha, Q_\beta\}|\psi\rangle$ for states in any SO(6) $\times$ SO(3) multiplet are given by two copies of the set

$$\Delta = H + \epsilon_1 \frac{\mu}{3} M^{12} + \epsilon_2 \frac{\mu}{6} M^{45} + \epsilon_3 \frac{\mu}{6} M^{67} - \epsilon_2 \epsilon_3 \frac{\mu}{6} M^{89}$$

where $\epsilon_i = \pm 1$ are chosen independently and M's are eigenvalues of the Cartan generators for the heighest weight state in the multiplet. Thus, BPS SO(6) $\times$ SO(3) multiplets are those for which

$$\lambda \cdot (1, \epsilon_1, \epsilon_2, \epsilon_3, -\epsilon_2\epsilon_3) = 0 \tag{7.2}$$

for one or more choices of ϵ_i, where

$$\lambda \equiv \left(3h, M^{12}, \frac{1}{2}M^{45}, \frac{1}{2}M^{67}, \frac{1}{2}M^{89}\right).$$

To relate this to our discussion in the rest of the paper, we should first determine the relation between this choice of Cartan generators and the Dynkin basis used in the rest of the paper. By explicitly writing the rotation generators in SU(4) $\times$ SU(2) notation, we find

$$\begin{aligned}
M^{12} &= \mathrm{diag}\left(0,0,0,0,\frac{1}{2},-\frac{1}{2}\right) = \frac{1}{2}H_5 \\
M^{45} &= \mathrm{diag}(1,1,-1,-1,0,0) = \frac{1}{2}(H_1 + 2H_2 + H_3) \\
M^{67} &= \mathrm{diag}(1,-1,1,-1,0,0) = \frac{1}{2}(H_1 + H_3) \\
M^{89} &= \mathrm{diag}(-1,1,1,-1,0,0) = \frac{1}{2}(H_3 - H_1) \\
3h &= \mathrm{diag}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}\right) = \frac{1}{4}H_1 + \frac{1}{2}H_2 + \frac{3}{4}H_3 + H_4 - \frac{1}{2}H_5
\end{aligned} \tag{7.3}$$

Now, starting with any SU(4|2) representation, we can consider any SU(4) $\times$ SU(2) representation in the decomposition, determine its highest weight in the Dynkin basis, calculate λ using eq. (7.3), and check whether the condition (7.2) is satisfied.

We first note that no typical representation can contain a BPS state, since the nearby typical representation with smaller a_4 would have a state with identical charges but smaller energy and therefore violate the BPS bound.

We next consider general atypical representations as depicted in figure 19. It is straightforward to verify that any SU(4) $\times$ SU(2) representation obtained by assigning all of the dark gray boxes (for supertableau with four or more rows) and either 0, 1, 2, or 3 of the light gray boxes to a fully symmetric SU(2) representation satisfy the condition (7.2) for at least one choice of ϵ_i. All other representations in the decomposition do not. For SU(2) non-singlet multiplets obtained in this way, the number of supercharges preserved (twice the number of choices of ϵ_i for which (7.2) is satisfied) is 8 minus twice the number of rows in the SU(4) tableau. For BPS SU(2) singlet multiplets, which arise only as the lowest energy multiplets for supertableaux with three or less rows (shown on the right in figure 19),

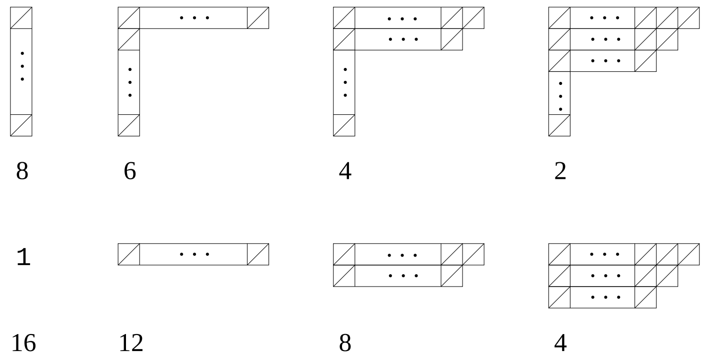

Figure 20: Maximum number of supersymmetries preserved among states of the atypical supermultiplets shown.

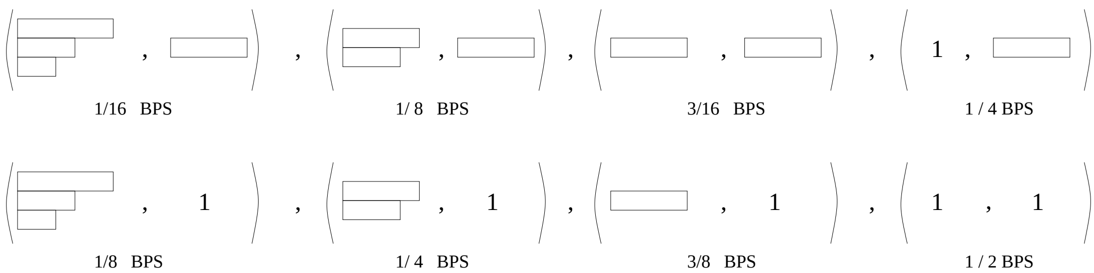

Figure 21: Fraction of SUSY preserved for BPS multiplets in atypical supermultiplet

the number of preserved supercharges is equal to 16 minus four times the number of rows in the SU(4) tableau.[17] The fraction of 32 supersymetries preserved for the various BPS $\mathrm{SU}(4) \times \mathrm{SU}(2)$ multiplets may be summarized as in figure 21.

Note that the representations shown will only be BPS if they have the appropriate energy, which is true only if they arise from an atypical multiplet in the manner depicted in figure 19.

For a given atypical $\mathrm{SU}(4|2)$ representation, we may then determine the $\mathrm{SU}(4) \times \mathrm{SU}(2)$ multiplet in its decomposition for which the largest number of supercharges is preserved. This maximum number is indicated in figure 20 for all possible atypical mutiplets. In cases where more than one of the tableaux apply (e.g. for the supertableaux with three vertical boxes which could be described both by the first and the last tableaux in figure 20) the maximum number of supercharges preserved is given by the largest number among the applicable tableaux. It is interesting to note that the tableaux in the lower row with "doubled" supersymmetry are those corresponding to the isolated point in figure 5 with $a_4 = a_5 = 0$. Finally, we note that the physical spectrum of the matrix model contains examples of all of these types of representations except for those with 12 supercharges preserved.

[17] The increased number of preserved supercharges for the special case of tableaux with less than four rows was not noted in the original version of this paper. We realized that this case must be treated separately upon reading the work of Kim and Park [13].

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We emphasize that even atypical multiplets containing states preserving 8 supercharges may also contain non-BPS states. A simple example is provided by the single oscillator states about the single-membrane vacuum of the SU(2) theory. These form a single atypical multiplet which includes states preserving eight (β_2), six ($\eta_{3/2}$), four (x_1), two ($\chi_{1/2}$), and 0 (α_0) supercharges. Thus, while α_0 is non-BPS and carries no charges whatsoever, it is part of an atypical multiplet and (as we have seen) its energy is protected.

8. Discussion

In this paper, we continued our analysis of the matrix model for M-theory on the maximally supersymmetric pp-wave background by exploring the physical consequences of the symmetry algebra. From its bosonic subalgebra $\mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{U}(1)_H$, we identified the symmetry algebra of the interacting $\mathrm{SU}(\mathbb{N})$ part of the theory to be the basic classical Lie superalgebra SU(4|2). Using the work of Kac [7], Bars et.al. [8] and Jakobsen [9], we described the complete set of physically allowable representations of SU(4|2). Among these are the typical multiplets, which lie on one-parameter families of representations differing only by their energy, and the atypical multiplets, for which there are no nearby multiplets with the same $\mathrm{SU}(4) \times \mathrm{SU}(2)$ state content but different energy. We argued that states in atypical multiplets can only receive energy corrections if two atypical multiplets combine into a typical multiplet, and found the complete set of multiplet pairs for which this is possible. Certain multiplets, including those known as doubly atypical do not appear in any of the pairs, and therefore are completely protected. Additional exactly protected quantities are provided by the supersymmetric indices given in eq. (4.3).

Equipped with this knowledge of the SU(4|2) representation theory, we turned to the actual spectrum of the matrix model for $\mu = \infty$, identified the complete set of SU(4|2) multiplets present, and used the representation theory reasoning to deduce which multiplets are protected as μ is made finite. We explicitly identified all doubly atypical (and therefore exactly protected) multiplets in the spectrum, and found that these include the vacuum states as well as infinite towers of excited states above the various vacua. Among the remaining atypical multiplets, we found some which are protected (either perturbatively or non-perturbatively) due to the absence of the complementary representations with which they could combine (either above the same vacuum or in the entire spectrum), and some which do combine and receive energy shifts (as verified by explicit perturbative calculation). Finally, we showed that the representation theory implies cancellations in the leading perturbative energy shifts for all states (typical and atypical) above the single-membrane vacuum, leaving a result that is finite in the large N limit.

By identifying protected multiplets in the matrix model for $\mu = \infty$ we have provided non-trivial information about the spectrum of the matrix model for all values of μ including the regime where perturbation theory is inapplicable. In particular, since the exactly

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protected doubly atypical spectrum about any given vacuum has a well defined large N limit, we may conclude that the states in this limiting spectrum (including the vacua themselves) are exact quantum states of M-theory on the pp-wave (assuming the validity of the Matrix Theory conjecture).

There are a number of interesting open questions and directions for future work.

One question is whether any of the atypical multiplets for the single-membrane vacuum receive an energy shift. We showed that this is prohibited by group theory for SU(2), but not for SU($\mathbb{N}$). On the other hand, in limited perturbative calculations, we did not find any atypical multiplet which received an energy shift, even in cases where the complementary atypical multiplet existed in the spectrum. It would be interesting to see if this holds beyond the leading order in perturbation theory and if so, to understand the underlying reason why these states do not combine. A related question is whether a state that is protected in the SU($\mathbb{N}_1$) theory for some N_1 is also protected in the SU($\mathbb{N}$) theory for higher N. We have seen some evidence for this through leading order perturbative calculations, but it would be interesting to understand whether or not this is true in general.

We have found that all states about the single membrane vacuum have energy shifts at leading order in perturbation theory which are finite in the large N limit, extending our results in [1] for the states $(a^\dagger_{00})^n|0\rangle$. A natural question is whether this finiteness of the perturbative corrections holds also to higher orders in perturbation theory. For example, one might evaluate the second energy correction for the lowest energy typical state $(a^\dagger_{00})^2|0\rangle$. It would be quite remarkable if the complete perturbative expansion were finite term-by-term for $N \to \infty$, since the supermembrane field theory is superficially quite non-renormalizable. Also, we showed in [1] that the height of the energy barrier between various vacua goes to zero in this limit. If the perturbation theory does turn out to be finite, we could use it to obtain reliable dynamical information about states in M-theory which are not protected by supersymmetry.

Finally, it would be interesting to understand the representation theory for the superalgebras corresponding to various string theories on other pp-wave backgrounds and see whether similar interesting protected multiplets exist in those cases. A more direct application of the analysis of this paper might be to the ordinary AdS/CFT conjecture, since the superalgebra we studied in this paper is a subalgebra of the $SU(2,2|4)$ superconformal algebra governing type IIB string theory on $AdS^5 \times S^5$.

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A. Supersymmetry generators in terms of modes at $\mu = \infty$

Explicit matrix expressions for the superalgebra generators were provided in [1] (appendix B). In particular, for the non-trivial supercharge, we have

$$Q_{I\alpha} = \sqrt{R}\,\mathrm{Tr}\Bigg(\left(\Pi^a - i\frac{\mu}{6R}X^a\right) \mathsf{g}^a_{IJ}\epsilon_{\alpha\beta}\psi^{\dagger J\beta} - \left(\Pi^i + i\frac{\mu}{3R}X^i\right)\sigma^{i\ \beta}_{\alpha}\psi_{I\beta} + \\ + \frac{1}{2}[X^i, X^j]\epsilon^{ijk}\sigma^{k\beta}_{\alpha}\psi_{I\beta} - \frac{i}{2}[X^a, X^b](\mathsf{g}^{ab})_I^{\ J}\psi_{J\alpha} + i[X^i, X^a]\sigma^{i\ \ \lambda}_{\alpha}\,\mathsf{g}^a_{IJ}\epsilon_{\lambda\beta}\psi^{\dagger J\beta}\Bigg)$$

where $\mathsf{g}^{ab} = \frac{1}{2}(\mathsf{g}^a\mathsf{g}^{\dagger b} - \mathsf{g}^b\mathsf{g}^{\dagger a})$ and the matrices g^a_{IJ} relate the antisymmetric product of two **4** representations of SU(4) to the vector of SO(6), with

$$\mathsf{g}^a(\mathsf{g}^b)^\dagger + \mathsf{g}^b(\mathsf{g}^a)^\dagger = 2\delta^{ab}\,.$$

Expanding X's about the vacuum solutions, i.e.

$$X^i = \frac{\mu}{3R}J^i + Y^i$$

choosing the gauge $A_0 = 0$, and making the proper rescalings

$$Y^i \to \sqrt{\frac{R}{\mu}}Y^i\,, \qquad X^a \to \sqrt{\frac{R}{\mu}}X^a\,, \qquad t \to \frac{1}{\mu}t$$

the supercharges $Q_{I\alpha}$ take the form

$$Q_{I\alpha} = Q^0_{I\alpha} + \left(\frac{R}{\mu}\right)^{\frac{3}{2}} Q^1_{I\alpha} \tag{A.1}$$

where

$$Q^0_{I\alpha} = \sqrt{\mu}\,\mathrm{Tr}\Bigg\{\left((\dot{X}^a - \frac{i}{6}X^a)\mathbf{1} + \frac{i}{3}[J^i, X^a]\sigma^i\right)_{\alpha}^{\ \beta}\mathsf{g}^a_{IJ}\epsilon_{\beta\lambda}\psi^{\dagger J\lambda} - \tag{A.2}$$

$$- \left(\dot{Y}^i + \frac{i}{3}Y^i - \frac{1}{3}[J^i, Y^j]\epsilon^{ijk}\right)\sigma^{i\ \ \beta}_{\ \alpha}\psi_{I\beta}\Bigg\} \tag{A.3}$$

and

$$Q^1_{I\alpha} = \sqrt{\mu}\,\mathrm{Tr}\left\{\frac{1}{2}[Y^i, Y^j]\epsilon^{ijk}\sigma^{i\ \ \beta}_{\ \alpha}\psi_{I\beta} - \frac{i}{2}[X^a, X^b]\mathsf{g}_{ab\ I}^{\ \ J}\psi_{J\alpha} + i[Y^i, X^a]\mathsf{g}^a_{\ IJ}(\sigma^i\epsilon)_{\alpha\beta}\psi^{\dagger J\beta}\right\} \tag{A.4}$$

In the $\mu = \infty$ limit, only the quadratic part Q^0 of the supercharge Q remains. One may explicitly check that

$$\begin{aligned}
\{Q^{\dagger\ 0I\alpha}, Q^0_{J\beta}\} &= \mu\left(2\delta^I_J\delta^\alpha_\beta H_2 - \frac{1}{3}\epsilon^{ijk}\sigma^{k\ \alpha}_{\ \beta}M_{ij} - \frac{i}{6}\delta^\alpha_{\ \beta}(\mathsf{g}^{ab})^I_{\ J}M_{ab}\right) \\
\{Q^{\dagger\ 0I\alpha}, Q^1_{J\beta}\} + \{Q^{\dagger\ 1I\alpha}, Q^0_{J\beta}\} &= 2\mu\delta^I_J\delta^\alpha_\beta H_3 \\
\{Q^{\dagger\ 1I\alpha}, Q^1_{J\beta}\} &= 2\mu\delta^I_J\delta^\alpha_\beta H_4
\end{aligned}$$

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Now let us concentrate on the Q^0 piece. Expanding X's in terms of the spherical harmonics and the eigen-modes about the irreducible $X = J$ vacuum we obtain[18]

$$
\begin{aligned}
Q_{I\alpha} = & \sum_{j=1}^{N-1} i\sqrt{\frac{2\mu}{3}} a_{j-1\ m} \begin{pmatrix} -\sqrt{j-m}\ (\chi^\dagger)_I{}^{j-\frac{1}{2}\ m-\frac{1}{2}} \\ -\sqrt{j+m}\ (\chi^\dagger)_I{}^{j-\frac{1}{2}\ m+\frac{1}{2}} \end{pmatrix}_\alpha \\
& + \sum_{j=1}^{N-1} i\sqrt{\frac{\mu}{3}} \mathrm{g}^a_{IJ} a^{a\ \dagger}_{jm} \begin{pmatrix} -\sqrt{j+m}\ (\chi^J)_{j-\frac{1}{2}\ -m-\frac{1}{2}} \\ -\sqrt{j-m}\ (\chi^J)_{j-\frac{1}{2}\ -m+\frac{1}{2}} \end{pmatrix}_\alpha \\
& + \sum_{j=0}^{N-1} i\sqrt{\frac{\mu}{3}} \mathrm{g}^a_{IJ} a^a_{jm} \begin{pmatrix} \sqrt{j-m+1}\ \eta^{\dagger\ J}_{j+\frac{1}{2}\ m-\frac{1}{2}} \\ -\sqrt{j+m+1}\ \eta^{\dagger\ J}_{j+\frac{1}{2}\ m+\frac{1}{2}} \end{pmatrix}_\alpha \\
& + \sum_{j=0}^{N-1} i\sqrt{\frac{2\mu}{3}} b^\dagger_{j+1\ m} \begin{pmatrix} \sqrt{j+m+1}\ \eta_I^{j+\frac{1}{2}\ -m-\frac{1}{2}} \\ -\sqrt{j-m+1}\ \eta_I^{j+\frac{1}{2}\ -m+\frac{1}{2}} \end{pmatrix}_\alpha .
\end{aligned}
$$

In the above a_{jm}, b_{jm} and a^a_{jm} are the annihilation operators for the α_{jm}, β_{jm} and x^a_{jm} modes, respectively. Note that sums for the last two lines are starting at $j = 0$. The multiplet generated through $j = 0$, which is of course in the U(1) part of the matrix model, is the only one particle, doubly atypical representation about $X = J$ vacuum.

For a general vacuum corresponding to M_i coincident membranes at radii N_i, we have a set of terms in Q corresponding to each ordered pair (N_i, N_j). For a given (N_i, N_j), the terms are exactly as above, except that the creation and annihilation operators are $M_i \times M_j$ and $M_j \times M_i$ matrices respectively, there is an overall trace, and the sums range over the spins listed in table 2.

B. Perturbative calculations

It was shown in [1] that the energy shift for the $X = 0$ vacuum state, i.e. the state $|0\rangle$ is zero (up to the second order in perturbation theory). Here we explicitly perform the calculation of the energy shift for the lowest non-trivial excited states above the $X = 0$ vacuum. The lowest energy non-trivial states (i.e. involving more than U(1) oscillators), are

$$
|\psi_{ab}\rangle = \frac{1}{\sqrt{(N^2-1)}} \left(\mathrm{Tr}(A^\dagger_a A^\dagger_b) - \frac{1}{N} \mathrm{Tr}(A^\dagger_a)\, \mathrm{Tr}(A^\dagger_b) \right) |0\rangle .
$$

This states have been normalized as

$$
\langle \psi_{ab} | \psi_{cd} \rangle = \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} \tag{B.1}
$$

and also they are chosen to be orthogonal to the states

$$
\mathrm{Tr}(A^\dagger_a)\, \mathrm{Tr}(A^\dagger_b)|0\rangle
$$

[18] We are using the notations of section 5 of [1] except for $\chi^\dagger$, which in our present notation $\chi^\dagger_{jm}$ creates a state with $J^3 = +m$.

which cannot receive any energy correction since they are in the U(1) part of the theory. Note that the states 21 states $|\psi_{ab}\rangle$ are degenerate and their energy is $\mu/3$. These 21 states can be put in $(1,1)$ and $(20,1)$ representations of $SU(2)\times SU(4)$ corresponding to contrating the SO(6) indices or symmetrizing them and subtracting off the trace (we previously computed the energy shift for the former state in [1].)

In order to do perturbative calculations, first we note that the cubic and quartic parts of the hamiltonian expanded about $X=0$ vacuum are [1]

$$\begin{aligned} H_3 = \left(\frac{R}{\mu}\right)^{\frac{3}{2}} \mathrm{Tr}\Bigg(& \sqrt{\frac{3}{8}} i\epsilon^{ijk}(A_iA_jA_k + 3A_i^\dagger A_jA_k + 3A_i^\dagger A_j^\dagger A_k + A_i^\dagger A_j^\dagger A_k^\dagger) - \\ & - \sqrt{\frac{3}{2}}\psi^{\dagger\alpha}\sigma^{i\,\beta}_{\alpha}[A_i + A_i^\dagger, \psi_\beta] - \\ & - \frac{\sqrt{3}}{2}\epsilon_{\alpha\beta}\psi^{\dagger\alpha}\mathrm{g}^a[A_a + A_a^\dagger, \psi^{\dagger\beta}] + \frac{\sqrt{3}}{2}\epsilon^{\alpha\beta}\psi_\alpha \mathrm{g}^{\dagger a}[A_a + A_a^\dagger, \psi_\beta]\Bigg) \end{aligned}$$

and

$$H_4 = -\left(\frac{R}{\mu}\right)^3 \mathrm{Tr}\left(\frac{1}{4}[X^i,X^j]^2 + \frac{1}{2}[X^i,X^a]^2 + \frac{1}{4}[X^a,X^b]^2\right)$$

It is easy to see that the energy shift at first order in perturbation theory

$$\langle\psi_{ab}|H_3|\psi_{cd}\rangle$$

will be zero for all states, since all terms in H_3 have non-zero H_2 eigenvalue. Thus H_3 acting on any state gives a combination of basis state all of whose energies are different than that of the original state.

At second order in perturbation theory, the energy shift is given by the eigenvalues of the 21×21 matrix

$$\begin{aligned} \Delta^{ab,cd} &= \Delta_4^{ab,cd} + \Delta_3^{ab,cd} \\ &= \langle\psi_{ab}|H_4|\psi_{cd}\rangle + \sum_n \frac{1}{E_\psi - E_n}\langle\psi_{ab}|H_3|n\rangle\langle n|H_3|\psi_{cd}\rangle \end{aligned} \tag{B.2}$$

Let us first focus on the $\Delta_3^{ab,cd}$ part. We note that H_3 may be written as a sum of terms H_3^i with specific H_2 eigenvalues,

$$H_3 = \sum H_3^i$$

such that

$$[H_2, H_3^i] = E_i H_3^i$$

where H_2 is the quadratic hamiltonian and all the E_i's are distinct.

Using this decomposition, we find [1]

$$\begin{aligned} \Delta_3^{ab,cd} &= \sum_n \frac{1}{E_\psi - E_n}\langle\psi_{ab}|H_3|n\rangle\langle n|H_3|\psi_{cd}\rangle \\ &= \sum_i -\frac{1}{E_i}\langle\psi_{ab}|(H_3^i)^\dagger H_3^i|\psi_{cd}\rangle \end{aligned}$$

Since the vacuum energy shift was zero, we can ignore all "disconnected" contributions in which the creation operators from the initial state contract only with annihilation operators from the final state since these terms will be the same as for the vacuum shift. A further simplification arises from the fact that the interaction vertices are written in terms of commutators of X's with the result that $\mathrm{Tr}(A^\dagger)$ or $\mathrm{Tr}(A)$ from the initial or final state contracted with any of the interaction terms will vanish. Thus, we may ignore the

$$-\frac{1}{N}\,\mathrm{Tr}(A_a^\dagger)\,\mathrm{Tr}(A_a^\dagger)$$

piece since it contributes only to the disconnected contributions. Since the calculations are very much similar to the one appeared in [1], here we skip the details of the calculations and only show the result,

$$\Delta_3^{ab,cd} = -216N\,(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\,.$$

Now let us work out the $\Delta_4^{ab,cd}$ piece. The connected part of Δ_4 also receives two contributions, coming from

$$H_4^1 = -\frac{1}{2}\,\mathrm{Tr}(X^aX^bX^aX^b - X^aX^aX^bX^b)$$

and

$$H_4^2 = -\,\mathrm{Tr}(X^aX^iX^aX^i - X^aX^aX^iX^i)\,.$$

Again using the tools introduced in [1] we find

$$\begin{aligned}\Delta_4^{1\ ab,cd} &= \langle\psi_{ab}|H_4^1|\psi_{cd}\rangle\\ &= \frac{1}{(N^2-1)}\cdot\frac{1}{2}\langle\mathrm{Tr}(A_aA_b)\,\mathrm{Tr}(X^eX^eX^fX^f - X^eX^fX^eX^f)\,\mathrm{Tr}(A_c^\dagger A_d^\dagger)\rangle\\ &= 18N\ \left[9(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}) + 2\delta_{ab}\delta_{cd}\right]\end{aligned}$$

and

$$\begin{aligned}\Delta_4^{2\ ab,cd} &= \langle\psi_{ab}|H_4^2|\psi_{cd}\rangle\\ &= \frac{1}{(N^2-1)}\langle\mathrm{Tr}(A_aA_b)\,\mathrm{Tr}(X^fX^fX^iX^i - X^fX^iX^fX^i)\,\mathrm{Tr}(A_c^\dagger A_d^\dagger)\rangle\\ &= 54N\ (\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})\end{aligned}$$

Putting the values if Δ_4 and Δ_3 together we find

$$\Delta^{ab,cd} = \Delta_3^{ab,cd} + \Delta_4^{1\ ab,cd} + \Delta_4^{2\ ab,cd} \tag{B.3}$$

$$= 36N\delta_{ab}\delta_{cd}\ \mu\left(\frac{R}{\mu}\right)^3\,. \tag{B.4}$$

Since $\Delta^{ab,cd}$ is already diagonal we can easily read off the energy shifts for $(1,1)$ and $(20,1)$ states. As we see the energy shift for $(20,1)$ is zero, while the energy shift for $(1,1)$ state is $36\times\frac{36}{12}N\mu\,(R/\mu)^3$, as we found in [1]. The factor of $1/12$ comes from properly normalizing the singlet part of the state (B.1) which we have used.

References

[1] K. Dasgupta, M.M. Sheikh-Jabbari and M. Van Raamsdonk, *Matrix perturbation theory for M-theory on a pp-wave*, *J. High Energy Phys.* **05** (2002) 056 [hep-th/0205185].

[2] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from* $N = 4$ *super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[3] R. Penrose, *Any space-time has a plane wave limit*, *Differential geometry and gravity*, pp. 271, Dordrecht 1976.

[4] R. Güven, *Plane wave limits and t-duality*, *Phys. Lett.* **B 482** (2000) 255 [hep-th/0005061].

[5] J. Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, *Phys. Lett.* **B 134** (1984) 194.

[6] J. Figueroa-O'Farrill and G. Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, *J. High Energy Phys.* **08** (2001) 036 [hep-th/0105308];
M. Blau, J. Figueroa-O'Farrill, C. Hull, G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081];
U. Gursoy, C. Núñez and M. Schvellinger, *Rg flows from spin(7), cy 4-fold and hk manifolds to AdS, Penrose limits and pp waves*, *J. High Energy Phys.* **06** (2002) 015 [hep-th/0203124].

[7] V.G. Kac, *Representations of classical Lie superalgebras*, in *Bonn 1977, proceedings, differential geometrical methods in mathematical physics 2*, Berlin 1977, pp. 597–626; *Lie superalgebras*, *Adv. Math.* **26** (1977) 8.

[8] I. Bars, *Supergroups and their representations*, *Lectures in Applied Mathematics*, **21** (1985) 17;
A. Baha Balantekin and I. Bars, *Dimension and character formulas for lie supergroups*, *J. Math. Phys.* **22** (1981) 1149; *Representations of supergroups*, *J. Math. Phys.* **22** (1981) 1810; *Branching rules for the supergroup* $\mathrm{SU}(N/M)$ *from those of* $\mathrm{SU}(N+M)$, *J. Math. Phys.* **23** (1982) 1239;
A. Baha Balantekin, *Anomalies and eigenvalues of casimir operators for lie groups and supergroups*, *J. Math. Phys.* **23** (1982) 486;
I. Bars, B. Morel and H. Ruegg, *Kac-Dynkin diagrams and supertableaux*, *J. Math. Phys.* **24** (1983) 2253.

[9] H.P. Jakobsen, *The full set of unitarizable highest weight modules of basic classical Lie superalgebras*, *Memoirs of the American Mathematical Society*, (1994) 532.

[10] N.-W. Kim and J. Plefka, *On the spectrum of pp-wave matrix theory*, hep-th/0207034.

[11] S.-J. Hyun and H.-J. Shin, *Branes from matrix theory in pp-wave background*, *Phys. Lett.* **B 543** (2002) 115 [hep-th/0206090];
K. Sugiyama and K. Yoshida, *Supermembrane on the pp-wave background*, hep-th/0206070; *BPS conditions of supermembrane on the pp-wave*, hep-th/0206132;
C. Sochichiu, *Continuum limit(s) of BMN matrix model: where is the (nonabelian) gauge group?*, hep-th/0206239;
D.-S. Bak, *Supersymmetric branes in pp wave background*, hep-th/0204033.

[12] *LiE, a computer algebra package for Lie group computations*, can be found at: http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE.

[13] N. Kim and J.-H. Park, *Superalgebra for M-theory on a pp-wave*, hep-th/0207061.